Stochastic analysis of point processes : beyond the Poisson process

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T H È S E

Contributions to stochastic analysis of point processes: Beyond the Poisson process

Ian Flint

Abstract

Determinantal point processes have sparked interest in very diverse fields, such as random matrix theory, point process theory, and networking. In this manuscript, we consider them as random point processes, i.e. a stochastic collection of points in a general space. Hence, we are granted access to a wide variety of tools from point process theory, which allows for a precise study of many of their probabilistic properties.

We begin by laying out the probabilistic framework under which we will work in the whole manuscript. We insist more particularly on advanced properties of the Papangelou conditional intensity which turns out to be central in the rest of our analysis and appears in numerous chapters of this manuscript.

Then, we move to the study of determinantal point processes from an applicative point of view. To that end, we propose different methods for their simulation in a very general setting. Moreover, we bring to light a series of models derived from the well-known Ginibre point process, which are quite suited for applications.

Thirdly, we introduce a differentiable gradient on the considered probability space. Thanks to some powerful tools from Dirichlet form theory, we discuss integration by parts for general point processes, and show the existence of the associated diffusion processes correctly associated to the point processes. We are able to apply these results to the specific case of determinantal point processes, which leads us to characterizing these diffusions in terms of stochastic differential equations.

Lastly, we turn our attention to the difference gradient on the same space. In a certain sense, we define a Skohorod integral, which satisfies an integration by parts formula, i.e. its adjoint is the difference operator. An application to the study of a random transformation of the point process is given, wherein we characterize the distribution of the transformed point process under mild hypotheses.

Table des matières

1 Résumé **7**

2 Introduction **33**

3 Introduction to the theory of point processes **43**
3.1 Locally finite point processes 43
3.2 Kernels and integral operators 46
3.3 α-determinantal point processes 48
3.4 Papangelou conditional intensity 52

4 Simulation of α-determinantal point processes **57**
4.1 Simulation of α-determinantal point processes by projection kernels 57
4.2 Perfect simulation of determinantal point processes 60
4.3 Definition of the Ginibre point process 62
4.4 Efficient simulation of the Ginibre point process 63
4.5 Summary 75

5 Stochastic analysis on the configuration space **77**
5.1 Introduction to Dirichlet forms 77
5.2 Differential gradient 86
5.3 Determinantal point process 88
5.4 General point process 103
5.5 Summary 110

6 Moment formulae **111**
6.1 Derivation of moment formulae 111
6.2 Extended moment formulae for the divergence 116
6.3 Random transformation of the point process 120
6.4 Summary 125

A Disaster recovery algorithm **127**

B Stochastic dynamics of determinantal processes by integration by parts **139**

C Perfect simulation of determinantal point processes **163**

Bibliography **175**

Chapitre 1

Résumé

Nous allons commencer par développer la théorie générale des processus ponctuels initiée dans [37], et le lecteur intéressé pourra aussi consulter [17, 18, 64].

Avant-propos

Les processus ponctuels ont été abondamment étudiés. Mais, dans l'écrasante majorité des cas, μ est un processus ponctuel de Poisson, dans quel cas la littérature est abondante. Toutefois, malgré ses qualités mathématiques et théoriques indéniables, le processus ponctuel de Poisson a aussi un désavantage considérable, celui de ne pas posséder de structure de corrélation. Pour s'en rendre compte, rappelons la manière dont le processus de Poisson est simulé. On commence par considérer un compact donné sur lequel va s'effectuer la simulation. Puis, on simule le nombre de points dans ce compact selon une loi de Poisson de paramètre la mesure de Lebesgue du compact. Enfin, on simule chacun des points uniformément sur le compact, et ce indépendamment des autres points. De sorte que le processus ponctuel de Poisson est rarement un modèle de simulation pertinent, puisque les phénomènes naturels que nous cherchons à modéliser possèdent souvent une structure de corrélation. Le problème auquel nous sommes alors confrontés est celui de la modélisation de ces corrélations, tout en se restreignant à une classe de processus ponctuels pour lesquels la théorie mathématique est suffisamment poussée.
Un processus ponctuel étant sans hypothèse supplémentaire très général, il nous faut restreindre un peu le cadre lorsqu'il s'agit de trouver un modèle pratique. Un exemple de classe de modèles pour la modélisation remonte à [58], et est connu sous le nom de processus ponctuel de Gibbs. Ce processus ponctuel est d'un intérêt majeur en physique théorique puisqu'il a une interprétation en terme de potentiels d'interaction entre les différents points, vus comme des particules. La théorie mathématique entourant les processus ponctuels de Gibbs est abondante, et ces processus ponctuels forment une classe très générale de processus ponctuels à répulsion.

2. Référence pour l'image : [34]

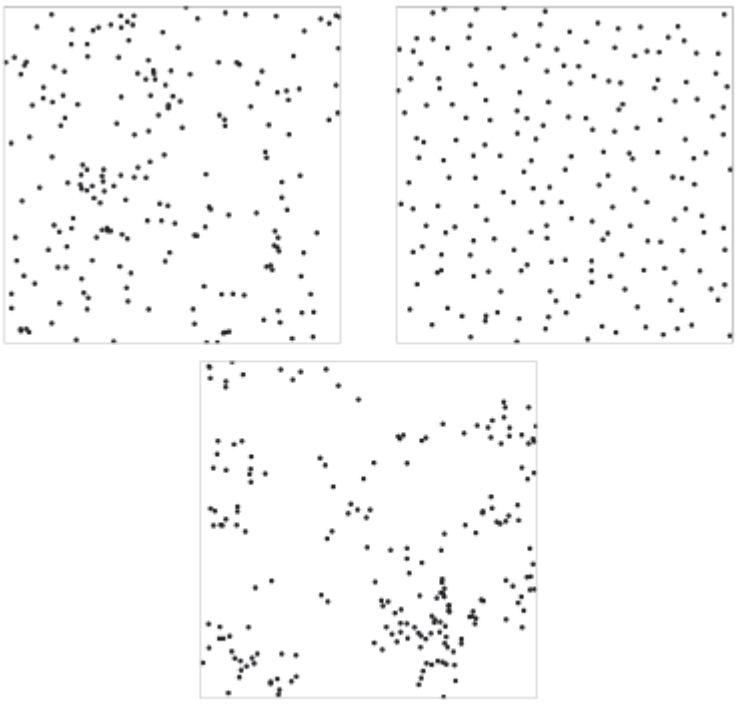


Figure 1.1 – Comparaison[2] du processus ponctuel de Poisson (en haut à gauche), du processus ponctuel déterminantal (en haut à droite), et du processus ponctuel permanental (en bas)

Les processus ponctuels déterminantaux ont été introduits par O. Macchi en 1975 afin de modéliser la position des fermions en physique théorique. Leurs analogues les processus ponctuels permanentaux, qui modélisent eux la position des bosons, sont en fait liés aux processus déterminantaux, comme l'ont montré T. Shirai et Y. Takahashi. Ces deux types de processus ponctuels font en fait partie d'une classe que nous allons appeler processus ponctuels α-déterminantaux. Ces processus ponctuels ont été étudié en détail dans [77], qui est une de nos références principales quant au cadre théorique permettant l'étude de ces processus.
Sans rentrer à ce stade dans le détail quant à la définition des processus α-déterminantaux, mentionnons que le type de processus ponctuel sera très différent selon le signe de $\alpha \in \mathbb{R}$. Spécifiquement, même si ces processus ponctuels ont été groupés dans [77], ils modéliseront soit des particules à répulsion (fermions), soit des particules à attraction (bosons), selon le signe de α. Cette différence fondamentale dans la structure de corrélation, et plus généralement dans le cadre probabiliste derrière chacun des processus ponctuels, nous amène dans ce manuscrit à ne considérer que le cas répulsif $\alpha < 0$. Même si un certain nombre de nos résultats pourraient être donnés dans un cadre général incorporant $\alpha \geq 0$, nous utiliserons abondamment le caractère répulsif du processus ponctuel, en particulier pour obtenir une domination stochastique par le processus ponctuel de Poisson. Ces résultats ne seront pas vérifiés lorsque $\alpha \geq 0$, comme on peut le voir dans [23] par exemple.
Nous donnons dans Figure 2.2 une illustration des propriétés répulsives d'un processus

ponctuel déterminantal, lorsqu'il est comparé au processus ponctuel de Poisson. Notons qu'il existe des processus ponctuels possédant plus de répulsion que le processus ponctuel déterminantal. Le paramètre α va quant à lui permettre d'ajuster la répulsion, et va servir de classe intermédiaire entre le processus ponctuel déterminantal (à droite dans Figure 2.2) et le processus ponctuel de Poisson (à gauche dans Figure 2.2).

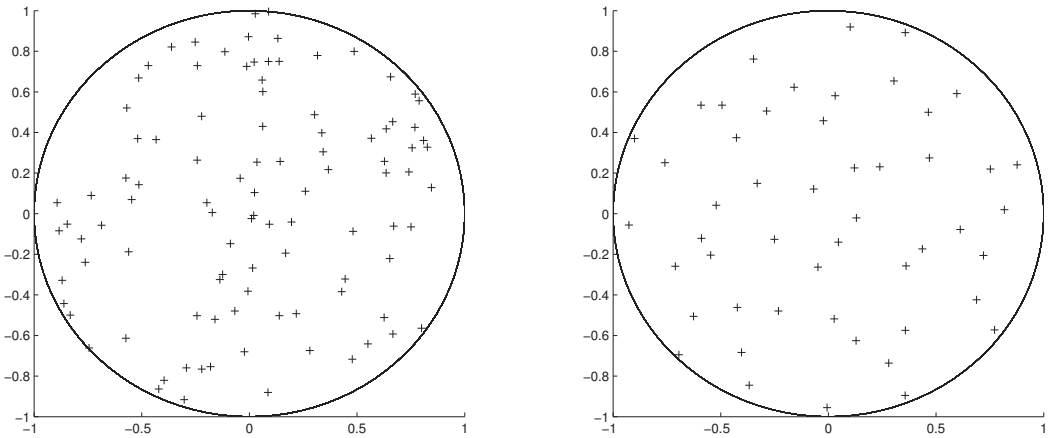
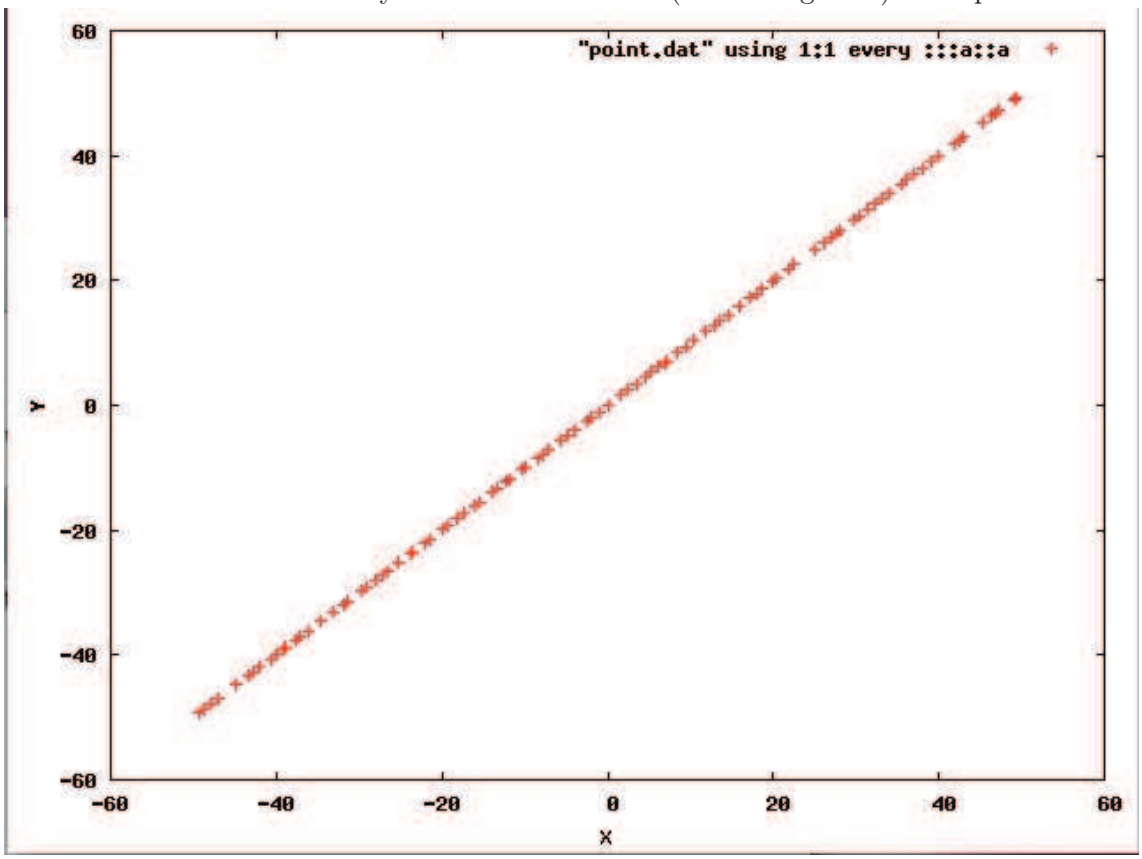


FIGURE 1.2 – Poisson vs determinantal

Nous présentons aussi dans Figure 2.3 un tirage du processus ponctuel de Dyson, qui est un processus ponctuel déterminantal sur $\mathbb{R}$ avec un noyau en forme de sinus cardinal. L'introduction du processus ponctuel de Dyson remonte à [22], voir aussi [81] pour des développement plus récents.

Figure 1.3 – Modèle de Dyson uni-dimensionnel (sur la diagonale) - 100 particules



Comme nous l'avons dit précédemment, les processus ponctuels déterminantaux modèlent à l'origine le comportement de particules physiques. Toutefois, ils sont apparus dans un domaine en apparence orthogonal : la théorie des matrices aléatoires. En effet, on s'en rendu compte que les valeurs propres d'une certaine classe de matrices aléatoires (par exemple, les matrices orthogonales tirées selon la mesure de Haaz) forme un processus ponctuel déterminantal.
Au contraire des processus déterminantaux, mentionnons que les processus ponctuels sont attractifs. Ils sont en fait une sous-classe de processus ponctuels bien connus sous le nom de processus de Cox. Les processus déterminantaux et permanentaux ont ainsi trouvé des application en finance, réseaux, épidémiologie, etc.
Dans ce manuscrit, nous nous intéresserons plus particulièrement au cas des processus α-déterminantaux, $\alpha < 0$. Dès que cela sera possible, nous généraliserons les résultats obtenus à un processus ponctuel très général, avec pour seule hypothèse l'existence d'une intensité de Papangelou (que nous définirons plus tard). Ces généralisations seront par exemple obtenues dans Section 5.4 ou Chapter 6.

Processus ponctuels

Soit E un espace polonais, et $\mathcal{O}(E)$ la famille des ouverts non-vides de E ainsi que $\mathcal{B}$ la tribu borélienne générée par $\mathcal{O}(E)$. Pour tout sous-ensemble $A \subset E$, on note $|A|$ le cardinal de l'ensemble A, avec la convention $|A| = \infty$ si A n'est pas fini. On utilise aussi la notation $\Lambda \subseteq E$, pour désigner un compact Λ inclus dans E. On note $\mathcal{X}$ l'ensemble des configurations localement finies de E, c'est-à-dire :

$$\mathcal{X} := \{\xi \subset E \ : \ |\xi \cap \Lambda| < \infty \quad \forall \Lambda \subseteq E\}.$$

En fait, $\mathcal{X}$ est en bijection avec l'ensemble des mesures positives *simples* de Radon sur E à valeurs entières (une mesure de Radon positive ξ est dite *simple* si pour tout $x \in E$, $\xi(x) \leq 1$). Parfois, on écrira $\mathcal{N}$ pour désigner l'ensemble des mesures de Radon positives (non-*simples*) à valeurs entières sur E. Ainsi, la topologie naturelle sur l'ensemble $\mathcal{X}$ est la topologie vague, qui est la topologie la plus faible telle que pour toutes les fonctions continues à support compact f sur E, l'application

$$\xi \mapsto \langle f, \xi \rangle := \sum_{y \in \xi} f(y)$$

est continue. La tribu correspondante sera notée $\mathcal{F}$, et sera définie par

$$\mathcal{F} := \sigma(\{\xi \in \mathcal{X} \ : \ |\xi \cap \Lambda| = m\}, \, m \in \mathbb{N}, \, \Lambda \subseteq E \ ).$$

Dans la suite, les éléments de $\mathcal{X}$ seront nommés de manière informelle configurations et on identifiera une configuration $\xi \in \mathcal{X}$ avec la mesure de Radon purement atomique $\sum_{y \in \xi} \varepsilon_y$, où ε_y désigne la mesure de Dirac en un point $y \in E$. Pour un $\xi = \sum_{y \in \xi} \varepsilon_y$, on considérera de la même manière que ξ peut être vu comme un ensemble, et on écrira $\xi \cup y_0 = \xi \cup \{y_0\}$ pour l'addition d'un point y_0 et $\xi \setminus y_0 = \xi \setminus \{y_0\}$ pour la soustraction d'un point y_0. On définit de manière similaire $\mathcal{X}_0$ l'ensemble des configurations finies sur E :

$$\mathcal{X}_0 := \{\xi \subset E : \ |\xi| < \infty\},$$

qui est équipé en tant que sous-ensemble de $\mathcal{X}$ par la tribu de trace $\mathcal{F}_0 = \mathcal{F}|_{\mathcal{X}_0}$. Enfin, pour tout compact $\Lambda \subseteq E$, on note $\mathcal{F}_\Lambda$ l'espace des configurations finies sur Λ, et $\mathcal{F}_\Lambda$ la tribu (de trace) associée.

Comme dans [28], on définit pour tout mesure de Radon λ sur E la mesure de ($\lambda-$)échantillonnage notée L^λ et définie sur $(\mathcal{X}_0, \mathcal{F}_0)$ par

$$\int_{\mathcal{X}_0} f(\alpha) \, L^\lambda(\mathrm{d}\alpha) := \sum_{n \geq 0} \frac{1}{n!} \int_{E^n} f(\{x_1, \ldots, x_n\}) \, \lambda(\mathrm{d}x_1) \ldots \lambda(\mathrm{d}x_n), \tag{1.1}$$

pour toute fonction $f : \mathrm{N}_f \to \mathbb{R}$ mesurable. De même, on définit la restriction de cette mesure à un compact $\Lambda \subseteq E$ par :

$$\int_{\mathcal{X}_0} f(\alpha) \, L^\lambda_\Lambda(\mathrm{d}\alpha) := \sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} f(\{x_1, \ldots, x_n\}) \, \lambda(\mathrm{d}x_1) \ldots \lambda(\mathrm{d}x_n),$$

pour toute fonction $f : \mathcal{X}_\Lambda \to \mathbb{R}$ mesurable.

Un processus ponctuel est par définition une mesure de probabilité μ sur l'espace mesurable $(\mathcal{X}, \mathcal{F})$. Une telle mesure de probabilité μ est caractérisée par sa transformée de Laplace $\mathcal{L}_\mu$, qui est définie pour toute fonction mesurable positive f sur E par

$$\mathcal{L}_\mu(f) = \int_{\mathcal{X}} e^{-\langle f,\xi\rangle}\, \mu(\mathrm{d}\xi).$$

Avec l'abus de notation classique, on notera de temps à autre ξ la variable aléatoire canonique sur l'espace probabilité $(\mathcal{X}, \mathcal{F}, \mu)$. Avec le même abus de notation, on notera $\mathbb{E}$ l'espérance de la variable aléatoire ξ, et on écrira ainsi

$$\mathbb{E}[F(\xi)] = \int_{\mathcal{X}} F(\xi)\, \mu(\mathrm{d}\xi),$$

pour toute fonction mesurable $F : \mathcal{X} \to \mathbb{R}$. On écrit

$$\xi_\Lambda = \xi \cap \Lambda$$

pour la restriction de ξ à un compact $\Lambda \subseteq E$. La loi de ξ_Λ (i.e. la restriction de μ à $\Lambda \subseteq E$) est notée μ_Λ. Enfin, pour un compact $\Lambda \subseteq E$, on note $\xi(\Lambda)$ le nombre de points de ξ_Λ, i.e. $\xi(\Lambda) := |\xi \cap \Lambda|$.

On dit qu'un processus ponctuel μ possède une fonction de corrélation $\rho : \mathcal{X}_0 \to \mathbb{R}$ par rapport à une mesure de Radon λ sur $(E, \mathcal{B})$ si ρ est mesurable et vérifie

$$\mathbb{E}[\sum_{\alpha\subset\xi,\ \alpha\in\mathcal{X}_0} f(\alpha)] = \int_{\mathcal{X}_0} f(\alpha)\, \rho(\alpha)\, L^\lambda(\mathrm{d}\alpha),$$

pour toute fonction mesurable positive f sur $\mathcal{X}_0$. Quand une telle mesure λ existe, on l'appelle la mesure sous-jacente. Pour une configuration finie $\alpha = \{x_1, \ldots, x_k\}$, où $k \in \mathbb{N}^*$, on écrit parfois $\rho(\alpha) = \rho_k(x_1, \ldots, x_k)$ et on dit que la fonction symétrique ρ_k sur E^k est la k-eme fonction de corrélation.

Proposition 1.0.1. *Les fonctions de corrélation de μ (si elles existent), par rapport à une mesure de Radon λ sur E, vérifient*

$$\mathbb{E}\left[\prod_{i=1}^{k} \xi(B_i)\right] = \int_{B_1\times\ldots\times B_k} \rho_k(x_1, \ldots, x_k)\, \lambda(\mathrm{d}x_1) \ldots \lambda(\mathrm{d}x_k),$$

pour toute famille de compacts mutuellement disjoints $B_1, \ldots, B_k$ de E, $k \geq 1$.

Démonstration. On définit u_N de la manière suivante :

$$u_N : \omega \mapsto \begin{cases} 1 \text{ si } |\omega| = N \text{ et il existe } x_1, \ldots, x_N \in \omega \text{ tels que } x_1 \in A_1, \ldots, x_N \in A_N, \\ 0 \text{ sinon .} \end{cases}$$

Et on obtient

$$\begin{aligned}\mathbb{E}[\xi(A_1)\dots\xi(A_N)] &= \int_{\mathcal{X}} \sum_{x_1,\dots,x_N\in\xi} \mathbf{1}_{A_1,\dots,A_N}(x_1,\dots,x_N)\,\mu(\mathrm{d}\xi)\\ &= \int_{\mathcal{X}} \sum_{\omega\in\mathcal{X}_0:\omega\subset\xi} u_N(\omega)\,\mu(\mathrm{d}\xi)\\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\int_{E^m} u_N(\{x_1,\dots,x_m\})\rho(\{x_1,\dots,x_m\})\,\lambda^{\otimes m}(\mathrm{d}x_1,\dots,\mathrm{d}x_m)\\ &= \int_{A_1\times\dots\times A_N}\rho(\{x_1,\dots,x_N\})\,\lambda^{\otimes m}(\mathrm{d}x_1,\dots,\mathrm{d}x_N).\end{aligned}$$

□

La formule de la Proposition 1.0.1 se généralise de la manière suivante :

Proposition 1.0.2. *Soient $A_1,\dots,A_n$ des compacts de E non-nécessairement disjoints. Soient $k_1,\dots,k_n$ des entiers vérifiant $\sum_{i=1}^n k_i = N$. Alors,*

$$\mathbb{E}[\prod_{i=1}^{n}\frac{\xi(A_i)!}{(\xi(A_i)-k_i)!}] = \int_{A_1^{k_1}\times\dots\times A_n^{k_n}}\rho(\{x_1,\dots,x_N\})\,\lambda(dx_1)\dots\lambda(dx_N).$$

Démonstration. La preuve se fait de la même manière que dans la proposition précédente. En effet, Proposition 1.0.2 est identique à Proposition 1.0.1 hormis le fait qu'il y a ici moins de compacts disjoints. □

On exigera de plus que $\rho_k(x_1,\dots,x_k) = 0$ dès lors que $x_i = x_j$ pour $1 \le i \ne j \le k$. Notons de plus que ρ_1 est la densité de points par rapport à λ, et

$$\rho_n(x_1,\dots,x_n)\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_n)$$

est donc la probabilité de trouver un point autour de chaque x_i, $i = 1,\dots,n$.

Pour tout compact $\Lambda \subseteq E$, les densités de Janossy de μ par rapport à une mesure de Radon λ sur E, sont (si elles existent) des fonctions mesurables $j_\Lambda^n : \Lambda^n \to \mathbb{R}$ telles que pour toutes fonctions mesurables $f : \mathcal{X}_\Lambda \to \mathbb{R}$,

$$\mathbb{E}\left[f(\xi_\Lambda)\right] = \sum_{n\ge 0}\frac{1}{n!}\int_{\Lambda^n} f(\{x_1,\dots,x_n\})\,j_\Lambda^n(x_1,\cdots,x_n)\;\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_n). \tag{1.2}$$

Remark 1. *Nous remarquons de plus que dans l'équation de définition (1.2), l'indice n n'apparaît pas directement. Ainsi, nous utiliserons aussi parfois la notation simplifiée $j_\Lambda(\alpha) := j_\Lambda^k(x_1,\dots,x_k)$, pour $\alpha = \{x_1,\dots,x_k\}$, où $k \in \mathbb{N}^*$.*

j_Λ est donc la densité de μ_Λ par rapport à L_Λ^λ, dès lors qur $\mu_\Lambda \ll L_\Lambda^\lambda$. Pour $n \ge 1$, les densités de Janossy satisfont les propriétés suivantes :

— Symétrie :

$$j_\Lambda^n\left(x_{\sigma(1)},\cdots,x_{\sigma(n)}\right) = j_\Lambda^n\left(x_1,\cdots,x_n\right),$$

pour toute permutation σ de $\{1,\cdots,n\}$.

— Contrainte de normalisation : pour tout compact $\Lambda \subseteq E$,

$$\sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\Lambda^n} j_\Lambda^n(x_1, \cdots, x_n)\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n) = 1.$$

Pour $n \geq 1$, la n-eme densité de Janossy $j_\Lambda^n(x_1, \dots, x_n)$ est en fait la densité jointe (à une constant près) des n points conditionnellement au fait que le processus ponctuel a exactement n points. Pour $n = 0$, $j_\Lambda^0(\emptyset)$ est la probabilité qu'il n'y ait pas de points dans Λ, et est donc parfois nommée probabilité de trous. On remarque aussi qu'il est possible de retrouver les densités de Janossy à partir des fonctions de corrélation à l'aide de la relation suivante :

$$j_\Lambda^n(x_1, \dots, x_n) = \sum_{m \geq 0} \frac{(-1)^m}{m!} \int_{\Lambda^m} \rho_{n+m}(x_1, \dots, x_n, y_1, \dots, y_m)\, \lambda(\mathrm{d}y_1) \dots \lambda(\mathrm{d}y_m),$$

pour tout compact $\Lambda \subseteq E$, et tous $x_1, \dots, x_n \in \Lambda$. La relation précédente est inversible et permet de manière similaire d'exprimer les fonctions de corrélation en fonction des densités de Janossy :

$$\rho_n(x_1, \dots, x_n) = \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} j_\Lambda^{m+n}(x_1, \dots, x_n, y_1, \dots, y_m)\, \lambda(\mathrm{d}y_1) \dots \lambda(\mathrm{d}y_m).$$

La preuve des deux équations précédentes se trouve par exemple dans [17].

Noyaux et opérateur intégral

À partir de maintenant, on se donne une mesure de Radon λ sur $(E, \mathcal{B})$. Pour tout compact $\Lambda \subseteq E$, on note $\mathrm{L}^2(\Lambda, \lambda)$ l'espace de Hilbert de fonctions de carré intégrable à valeurs complexes par rapport à la restriction à Λ de la mesure de Radon λ, équipé du produit scalaire

$$< f, g >_{\mathrm{L}^2(\Lambda,\lambda)} := \int_\Lambda f(x)\overline{g(x)}\, \lambda(\mathrm{d}x), \quad f, g \in \mathrm{L}^2(\Lambda, \lambda)$$

où ici $\overline{z}$ est le conjugué d'un complexe $z \in \mathbb{C}$. Par définition, un noyau K est une fonction mesurable de E^2 dans $\mathbb{C}$. On dit que K est localement de carré intégrable, si pour tout compact $\Lambda \subseteq E$, on a

$$\int_{\Lambda^2} |K(x, y)|^2 \lambda(\mathrm{d}x)\lambda(\mathrm{d}y) < \infty.$$

À tout noyau localement de carré intégrable K, on associe l'opérateur intégral

$$\mathcal{K}_\Lambda : \mathrm{L}^2(\Lambda, \lambda) \to \mathrm{L}^2(\Lambda, \lambda),$$

où Λ est un compact de E, défini par

$$\mathcal{K}_\Lambda f(x) := \int_\Lambda K(x, y) f(y)\, \lambda(\mathrm{d}y), \quad \text{for } \lambda\text{-almost all } x \in \Lambda.$$

Une application directe de l'inégalité de Cauchy-Schwarz montre que l'opérateur $\mathcal{K}_\Lambda$ est borné quand le noyau K est localement de carré intégrable. En fait, on peut montrer que

$\mathcal{K}_\Lambda$ est un opérateur compact.

À tout noyau localement de carré intégrable K, on associe l'opérateur intégral $\mathcal{K}$ défini par

$$\mathcal{K}f(x) := \int_E K(x,y)f(y)\,\lambda(\mathrm{d}y), \quad \text{for } \lambda\text{-almost all } x \in E$$

pour toutes fonctions $f \in \mathrm{L}^2(E,\lambda)$ ayant pour support un compact de E. On note $\mathcal{P}_\Lambda$ l'opérateur de projection de $\mathrm{L}^2(E,\lambda)$ dans $\mathrm{L}^2(\Lambda,\lambda)$. Dans ce cas, l'opérateur $\mathcal{K}_\Lambda$ défini précédemment est la projection $\mathcal{K}$ sur $\mathrm{L}^2(\Lambda,\lambda)$, ou de manière équivalente $\mathcal{K}_\Lambda = \mathcal{P}_\Lambda\mathcal{K}\,\mathcal{P}_\Lambda$. On notera aussi K_Λ le noyau de $\mathcal{K}_\Lambda$, i.e. $K_\Lambda(x,y) := \mathbf{1}_\Lambda(x)K(x,y)\mathbf{1}_\Lambda(y)$, pour $x,y \in E$. On dit que l'opérateur$\mathcal{K}$ est hermitien ou auto-adjoint si son noyau vérifie

$$K(x,y) = \overline{K(y,x)}, \quad \text{for } \lambda^{\otimes 2}\text{-almost all } (x,y) \in E^2. \tag{1.3}$$

De manière équivalente, cela signifie que l'opérateur intégral associé $\mathcal{K}_\Lambda$ est auto-adjoint pour tout compact $\Lambda \subseteq E$. Si $\mathcal{K}_\Lambda$ est auto-adjoint, par le théorème spectral pour des opérateurs auto-adjoints compacts, on a que $\mathrm{L}^2(\Lambda,\lambda)$ possède une base orthonormale $\{\varphi_j^\Lambda\}_{j\geq 1}$ composée de vecteurs propres de $\mathcal{K}_\Lambda$. Les valeurs propres correspondantes $\{\lambda_j^\Lambda\}_{j\geq 1}$ ont une multiplicité finie (sauf potentiellement la valeur proper 0) et l'unique point d'accumulation possible de valeurs propres est 0. Alors, le noyau K_Λ de $\mathcal{K}_\Lambda$ peut être décomposé dans cette base :

$$K_\Lambda(x,y) = \sum_{n\geq 1} \lambda_n^\Lambda \varphi_n^\Lambda(x)\overline{\varphi_n^\Lambda(y)}, \tag{1.4}$$

pour $x,y \in \Lambda$. On dit qu'un opérateur $\mathcal{K}$ set strictement positif (respectivement positif) si son spectre est inclus dans $(0,+\infty)$ (respectivement $[0,+\infty)$). Pour deux opérateurs $\mathcal{K}$ et $\mathcal{I}$, on dit que $\mathcal{K} > \mathcal{I}$ (respectivement $\mathcal{K} \geq \mathcal{I}$) si $\mathcal{K} - \mathcal{I}$ est un opérateur strictement positif (respectivement un opérateur positif).

On dit qu'un opérateur auto-adjoint intégral $\mathcal{K}_\Lambda$ est à trace si

$$\|\mathcal{K}_\Lambda\|_1 := \sum_{n\geq 1} |\lambda_\Lambda^n| < \infty,$$

et on note $\|\mathcal{K}_\Lambda\|_1$ la norme de trace de $\mathcal{K}_\Lambda$. On définit alors la trace de l'opérateur $\mathcal{K}_\Lambda$ par $\mathrm{Tr}\,\mathcal{K}_\Lambda = \sum_{n\geq 1}\lambda_\Lambda^n$. Si $\mathcal{K}_\Lambda$ est à trace pour tout compact $\Lambda \subseteq E$, alors on dit que $\mathcal{K}$ est localement à trace. On remarque facilement que si on opérateur hermitien intégral $\mathcal{K} : \mathrm{L}^2(E,\lambda) \to \mathrm{L}^2(E,\lambda)$ est à trace, alors $\mathcal{K}^n$ est aussi à trace, pour $n \geq 2$. En fait, on a même l'inégalité $\mathrm{Tr}(\mathcal{K}^n) \leq \|\mathcal{K}\|^{n-1}\mathrm{Tr}(\mathcal{K})$, où $\|\mathcal{K}\|$ est la norme opérateur de $\mathcal{K}$.

En pratiqué, pour évaluer des puissances fractionnaires de déterminants de Fredholm, on aura besoin d'introduire la notion d'α-determinants. Prenons donc $\alpha \leq 0$. Pour une matrice carrée $A = (A_{ij})_{1\leq i,j\leq n}$ de taille $n \times n$, l'α-determinant $\det_\alpha A$ de A est défini par :

$$\det{}_\alpha A = \sum_{\sigma\in S_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^n A_{i\sigma(i)}, \tag{1.5}$$

où S_n est le n-eme groupe symétrique et $\nu(\sigma)$ est le hombre de cycles de la permutation $\sigma \in S_n$. La définition précédente étend en fait celle de déterminant, qui est ici obtenue en

prenant $\alpha = -1$. Maintenant, il est possible de définir le déterminant de Fredholm de $\mathrm{I}+\mathcal{K}$ par

$$\mathrm{Det}(\mathrm{I}+\mathcal{K}) = \exp\left(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\mathrm{Tr}(\mathcal{K}^n)\right), \tag{1.6}$$

puisque $\mathrm{Tr}(\mathcal{K}^n) < \infty$. Ici, I dénote l'identité de $\mathrm{L}^2(E,\lambda)$ et $\mathcal{K}$ est un opérateur à trace général de $\mathrm{L}^2(E,\lambda)$. Alors, les puissances fractionnaires de déterminants de Fredholm peuvent être calculées comme suit. Pour tout opérateur intégral à trace $\mathcal{K}$, on a

$$\mathrm{Det}(\mathrm{I}-\alpha\mathcal{K})^{-1/\alpha} = \sum_{n\geq 0}\frac{1}{n!}\int_{E^n}\det_\alpha(K(x_i,x_j))_{1\leq i,j\leq n}\,\lambda(\mathrm{d}x_1)\ldots\lambda(\mathrm{d}x_n), \tag{1.7}$$

où K est le noyau de $\mathcal{K}$ et $|\alpha| \leq 1$. (1.7) a été obtenu dans le Théorème 2.4 de [77].

Enfin, terminons cette section en rappelant le résultat suivant de [28, Lemma A.4] :

Proposition 1.0.3. *Soit $\mathcal{K}$ un opérateur intégral positif, borné, et localement à trace de $\mathrm{L}^2(E,\lambda)$. Alors on peut choisir son noyau K (défini partout) tel que les propriétés suivantes sont vérifiées*

(i) K est positif, au sens que pour tous $c_1,\ldots,c_n \in \mathbb{C}$, et presque tous $x_1,\ldots,x_n \in E$, on a $\sum_{i,j=1}^n \bar{c}_i K(x_i,x_j)c_j \geq 0$.

(ii) K est un opérateur de Carlman, i.e. $K_x = K(\cdot,x) \in \mathrm{L}^2(E,\lambda)$ pour presque tout $x \in E$.

(iii) Pour tout compact $\Lambda \subseteq E$, $\mathrm{Tr}\,\mathcal{K}_\Lambda = \int_\Lambda K(x,x)\,\lambda(\mathrm{d}x)$ et

$$\mathrm{Tr}\,(\mathcal{P}_\Lambda\mathcal{K}^k\mathcal{P}_\Lambda) = \int_\Lambda < K_x, \mathcal{K}^{k-2}K_x >_{\mathrm{L}^2(\Lambda,\lambda)}\ \lambda(\mathrm{d}x),$$

pour $k \geq 2$.

Donc, sous certaines hypothèses sur $\mathcal{K}$, son noyau peut être choisi selon la proposition précédente.

α-determinantal point processes

Soit

$$\mathbf{M} = \{\alpha \leq 0\ :\ \exists m \in \mathbb{N},\ \alpha = -\frac{1}{m+1}\} \cup \{0\}.$$

En gardant à l'esprit les définitions précédentes, nous allons maintenant définir des processus ponctuels α-determinantaux. Pour ce faire, on introduit l'hypothèse suivante :

Hypothesis 1. *Supposons que $\alpha \in \mathbf{M}$. De plus, supposons que l'application $\mathcal{K}$ est un opérateur de Hilbert-Schmidt de $\mathrm{L}^2(E,\,\lambda)$ dans $\mathrm{L}^2(E,\,\lambda)$ qui satisfait les conditions suivantes :*

i) $\mathcal{K}$ est un opérateur intégral borné et auto-adjoint $\mathrm{L}^2(E,\,\lambda)$, avec noyau $K(.,.)$.

ii) Le spectre de $\mathcal{K}$ est inclus dans $[0,\,-\frac{1}{\alpha}]$, i.e. $0 \leq \mathcal{K} < -\frac{1}{\alpha}\,\mathrm{I}$.

iii) L'application $\mathcal{K}$ est localement à trace.

Supposons que $\mathcal{K}$ satisfait Hypothesis 3. Un processus ponctuel *simple* μ sur E est appelé un processus ponctuel α-determinantal process si ses fonctions de corrélations par rapport à une mesure de Radon donnée λ sur $(E, \mathcal{B})$ existent et satisfont

$$\rho_k(x_1, \dots, x_k) = \det_\alpha(K(x_i, x_j))_{1 \le i,j \le k},$$

pour tous $k \geq 1$ et $x_1, \dots, x_k \in E$, et où l'α-determinant a été défini en (1.5). Quand $\alpha = -1$, on dit que μ est un processus ponctuel déterminantal, et dans ce cas l'α-determinant est en fait le déterminant usuel. Quand $\alpha \neq -1$, μ est appelé un processus ponctuel α-determinantal. On remarque de plus que sous Hypothesis 1, on peut choisir $\mathcal{K}$ de sorte qu'il vérifie les points de 1.0.3. Clairement, nous aurons alors $\rho_k(x_1, \dots, x_k) \geq 0$ pour λ-presque tout $x_1, \dots, x_k \in E$.
En général, on écrit $\mu^\alpha_{\mathcal{K},\lambda}$ pour le processus ponctuel α-determinantal avec un opérateur intégral $\mathcal{K}$ et une mesure sous-jacente λ sur E. On omet l'indice α pour désigner le processus ponctuel déterminantal d'opérateur intégral $\mathcal{K}$ et de mesure sous-jacente λ.

L'existence et l'unicité (en loi) des processus ponctuels déterminantaux (ici, $\alpha = -1$) est assurée sous Hypothesis 3 par les résultats de [53], [77] et [80]. Voir aussi Lemme 4.2.6 et Théorème 4.5.5 dans [35]. Plus précisément, si un noyau K et l'opérateur intégral associé $\mathcal{K}$ satisfont Hypothesis 3, alors il existe un processus ponctuel déterminantal $\mu_{\mathcal{K},\lambda}$ sur E associé à $\mathcal{K}$. De plus, pour tout compact $\Lambda \subseteq E$ il existe des constantes $c_1(\Lambda), c_2(\Lambda) > 0$ telles que $\mu_{\mathcal{K},\lambda}(\xi(\Lambda) > k) \leq c_1(\Lambda)\mathrm{e}^{-c_2(\Lambda)k}$ pour tout $k \geq 1$, et dans ce cas les fonctions de corrélation $\rho_k(x_1, \dots, x_k)$ déterminent de manière unique la loi du processus ponctuel. Ceci est dû au fait que pour des compacts disjoints $B_1, \dots, B_k \subseteq E$, le vecteur aléatoire $(\xi(B_1), \dots, \xi(B_k))$ a une transformée de Laplace convergente dans un voisinage de 0 si les queues de la distribution de $\xi(\Lambda)$ sont exponentielles, c.f. [35, Remark 1.2.4]. Une autre manière de prouver l'unicité du processus ponctuel résultant est d'utiliser le critère général donné dans [45] et qui assure l'unicité :

$$\sum_{k \geq 0} \Big(\frac{1}{k!} \int_{\Lambda^k} \rho_k(x_1, \dots, x_k)\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_k) \Big)^{-1/k} = +\infty,$$

pour tout compact $\Lambda \subseteq E$. Afin de prouver que la série précédente diverge, on peut utiliser la formule classique de Fredholm (voir [79]) qui dit que

$$\frac{1}{k!} \int_{\Lambda^k} \det(K(x_i, x_j))_{1 \le i,j \le k}\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_k) = \mathrm{Tr}\,(\wedge^k \mathcal{K}_\Lambda),$$

où on a défini $\wedge^k \mathcal{T} := \mathcal{T} \otimes \dots \otimes \mathcal{T}|_{\mathcal{A}_s(\mathrm{L}^2(E,\lambda)^{\otimes k})}$, pour tout opérateur $\mathcal{T}$ on $\mathrm{L}^2(E, \lambda)$, et où $\mathcal{A}_s(H)$, pour H un espace de Hilbert séparable, est son sous-espace anti-symétrique. Alors, il suffit d'utiliser l'inégalité $\mathrm{Tr}\,(\wedge^k \mathcal{K}_\Lambda) \leq \frac{1}{k!}\mathrm{Tr}\,(\wedge \mathcal{K}_\Lambda)^k$ qui est prouvée par exemple dans [79]. On obtient en somme

$$\Big(\frac{1}{k!} \int_{\Lambda^k} \rho_k(x_1, \dots, x_k)\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_j) \Big)^{-1/k} \geq (k!)^{1/k} \mathrm{Tr}\,(\wedge \mathcal{K}_\Lambda)^{-1} \xrightarrow[k \to \infty]{} +\infty,$$

ce qui prouve l'unicité.

Pour $\alpha \in \mathbf{M}$, l'existence et l'unicité des processus ponctuels α-déterminantaux est garantie sous Hypothesis 1 par les résultats de [77] pour des processus ponctuels α-déterminantaux,

voir aussi [53] et [80] pour le cas déterminantal. Essentiellement, l'existence et l'unicité est obtenu par consistance des fonctions de Janossy. Par contre, il a été prouvé dans [8] que pour $\alpha \notin \mathbf{M}$, les processus ponctuels α-déterminantaux n'existent pas pour tout noyau $\mathcal{K}$. Plus précisément, il est obtenu dans [8] le résultat suivant :

Theorem 1.0.1. *Pour $\alpha \notin \mathbf{M}$, on n'a pas que pour toute matrice réelle symétrique positive et semi-définie A,*

$$\det_\alpha A \geq 0,$$

et en conséquence, les fonctions de corrélation ne sont pas correctement définies pour $\alpha \notin \mathbf{M}$.

À partir de maintenant, on considère donc $\alpha \in \mathbf{M}$, et on suppose que l'opérateur intégral $\mathcal{K}$ satisfait Hypothesis 1. Rappelons maintenant le résultat de [77] qui donne la transformée de Laplace de $\mu^\alpha_{\mathcal{K},\lambda}$.

Theorem 1.0.2. *Soit $\mathcal{K}$ un opérateur satisfaisant Hypothesis 1. Alors, l'unique processus ponctuel α-déterminantal $\mu^\alpha_{\mathcal{K},\lambda}$ a pour transformée de Laplace :*

$$\mathcal{L}_{\mu^\alpha_{\mathcal{K},\lambda}}(f) = \mathrm{Det}\left(\mathrm{I} + \alpha \mathcal{K}_\varphi\right)^{-1/\alpha},$$

pour toute fonction positive f sur E avec support compact, où $\varphi = 1 - e^{-f}$ et $\mathcal{K}_\varphi$ est l'opérateur intégral à trace, avec noyau

$$K_\varphi(x,y) = \sqrt{\varphi(x)} K(x,y) \sqrt{\varphi(y)}, \quad x, y \in E.$$

De plus, $\mu^\alpha_{\mathcal{K},\lambda}$ est nécessairement simple.

En corollaire du théorème précédent, on obtient que $\mu^\alpha_{\mathcal{K},\lambda}$ est la superposition de $m+1$ copies indépendantes d'un processus ponctuel déterminantal de noyau $\frac{\mathcal{K}}{m+1}$ et de mesure sous-jacente λ.

Soit $\mathcal{K}$ un opérateur satisfaisant Hypothesis 1. On définit l'opérateur intégral à trace sur $\mathrm{L}^2(E,\lambda)$ par

$$\mathcal{J}[\Lambda] := (\mathrm{I} + \alpha \mathcal{K}_\Lambda)^{-1} \mathcal{K}_\Lambda,$$

où le compact $\Lambda \subseteq E$ indexe les opérateurs $\mathcal{J}[\Lambda]$. $\mathcal{J}[\Lambda]$ est défini de façon à ce que $\mathcal{K}$ et $\mathcal{J}[\Lambda]$ soient quasi-inverses, i.e. que

$$(\mathrm{I} + \alpha \mathcal{K}_\Lambda)\,(\mathrm{I} - \alpha \mathcal{J}[\Lambda]) = \mathrm{I}\,. \tag{1.8}$$

L'opérateur $\mathcal{J}[\Lambda]$ est appelé opérateur d'interaction local et on insiste sur le fait que $\mathcal{K}_\Lambda$, $\mathcal{J}[\Lambda]$ n'est pas un opérateur de projection, i.e. en général, $\mathcal{J}[\Lambda] \neq \mathcal{P}_\Lambda (\mathrm{I} + \alpha \mathcal{K})^{-1} \mathcal{K} \mathcal{P}_\Lambda$. Malgré tout, $\mathcal{J}[\Lambda]$ possède un certain nombre de propriétés remarquables résumées dans [28]. On en rappelle ici quelques-unes qui seront utiles dans notre contexte. Tout d'abord, il est immédiat que $\mathcal{J}[\Lambda]$ existe en tant qu'opérateur borné puisque $\|\alpha\mathcal{K}\| < 1$, et son spectre est inclus dans $[0, +\infty)$. $\mathcal{J}[\Lambda]$ est encore un opérateur intégral, notons donc $J[\Lambda]$ son noyau (en fait, on peut même montrer que $\mathcal{J}[\Lambda]$ est aussi un opérateur de Carlman, c.f. [28]). De plus, comme $\mathcal{J}[\Lambda] \leq (1 + \|\alpha\mathcal{K}\|)^{-1} \mathcal{K}_\Lambda$, on a que $\mathcal{J}[\Lambda]$ est encore un opérateur à trace. Pour $\alpha = \{x_1, \ldots, x_k\} \in \mathcal{X}_\Lambda$, on note $\det_\alpha J[\Lambda](\alpha)$ $(= \det_\alpha J[\Lambda](\{x_1, \ldots, x_k\}))$ le déterminant $\det_\alpha \left(J[\Lambda](x_i, x_j)\right)_{1 \leq i,j \leq k}$. Notons que pour tout $k \in \mathbb{N}^*$, la fonction

$$(x_1, \ldots, x_k) \mapsto \det_\alpha J[\Lambda](\{x_1, \ldots, x_k\})$$

est $\lambda^{\otimes k}$-p.s. positive (grâce à Proposition 1.0.3) et symétrique en $x_1, \dots, x_k$ (voir e.g. [28]), et on écrira simplement $\det_\alpha J[\Lambda](\{x_1, \dots, x_k\}) = \det_\alpha J[\Lambda](x_1, \dots, x_k)$. L'opérateur d'interaction local est particulièrement central dans l'étude des processus ponctuels α-déterminantaux à cause de la proposition suivante :

Proposition 1.0.4 ([77]). *Supposons que le noyau $\mathcal{K}$ satisfasse Hypothesis 1. Alors, le processus ponctuel α-déterminantal $\mu^\alpha_{\mathcal{K},\lambda}$ possède des densités de Janossy j^n_Λ, et celles-ci sont données pour un compact $\Lambda \subseteq E$ et $n \in \mathbb{N}^*$ par*

$$j^n_\Lambda(x_1, \dots, x_n) = \mathrm{Det}(\mathrm{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \det_\alpha J[\Lambda](x_1, \dots, x_n), \tag{1.9}$$

pour $x_1, \dots, x_n \in \Lambda$. On peut aussi calculer la probabilité de vide $j^0_\Lambda(\emptyset) = \mathrm{Det}(\mathrm{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha}$. Ici, Det *est le déterminant de Fredholm défini en* (1.6).

Ici, on peut remarquer que (1.9) a encore du sens si $\|\mathcal{K}_\Lambda\| = 1$, puisque les zéros de $\mathrm{Det}(\mathrm{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha}$ sont du même ordre que les pôles de $\det_\alpha J[\Lambda](x_1, \dots, x_k)$, ces deux fonctions étant vues comme des fonctions de $\lambda^1_\Lambda, \dots, \lambda^n_\Lambda, \dots$ les valeurs propres de $\mathcal{K}_\Lambda$. Cette intuition est prouvée dans [77]. Il est aussi possible de calculer la fonction génératrice des moments d'un processus ponctuel α-déterminantal.

Proposition 1.0.5 (Fonction génératrice des moments d'un processus ponctuel α-déterminantal)**.** *La fonction génératrice des moments du nombre de points dans un compact $\Lambda \subseteq E$ est*

$$\mathrm{M}(u) = \mathbb{E}[e^{u\xi(\Lambda)}] = \mathrm{Det}(\mathrm{I} + \alpha(1 - e^u)K_\Lambda)^{-1/\alpha},$$

pour $u \in \mathbb{R}$.

Démonstration. Rappelons que pour une fonction mesurable $F : \mathbb{N} \to \mathbb{R}_+$,

$$\int_{\mathcal{X}_\Lambda} F(\xi(\Lambda))\, \mu(\mathrm{d}\xi) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} F(m) j_\Lambda(\{x_1, \dots, x_m\}) \lambda^{\otimes m}(\mathrm{d}x_1, \dots, \mathrm{d}x_m).$$

Dans ce cas, par la formule (1.9), on a

$$\mathrm{M}(u) = \mathrm{Det}(\mathrm{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \det_\alpha(e^u J[\Lambda](x_1, \dots, x_m)) \lambda^{\otimes m}(\mathrm{d}x_1, \dots, \mathrm{d}x_m).$$

Ici, il reste à reconnaître le développement de $\mathrm{Det}(\mathrm{I} - \alpha e^u \mathcal{J}[\Lambda])^{-1/\alpha}$ prouvé dans la formule (3.7), et on obtient

$$\mathrm{M}(u) = \mathrm{Det}\left((\mathrm{I} + \alpha\mathcal{K}_\Lambda)(\mathrm{I} - \alpha e^u \mathcal{J}[\Lambda])\right)^{-1/\alpha}.$$

En développant le produit, on a par (1.8)

$$(\mathrm{I} + \alpha\mathcal{K}_\Lambda)(\mathrm{I} - \alpha e^u \mathcal{J}[\Lambda]) = \mathrm{I} + (\mathrm{I} + \alpha\mathcal{K}_\Lambda)\left(\alpha\mathcal{J}[\Lambda](1 - e^u)\right) = \mathrm{I} + \alpha(1 - e^u)\mathcal{K}_\Lambda,$$

ce qui conclut la preuve. □

Remarquons que $J[\Lambda]$ peut être décomposé dans la même base de $\mathrm{L}^2(\Lambda, \lambda)$ que K_Λ. Plus précisément, en conséquence de (1.4), on obtient

$$J[\Lambda](x, y) = \sum_{n \geq 1} \frac{\lambda^n_\Lambda}{1 + \alpha\lambda^n_\Lambda} \varphi^n_\Lambda(x) \overline{\varphi^n_\Lambda(y)}, \tag{1.10}$$

pour $x, y \in \Lambda$.

Enfin, donnons des propriétés faisant le lien entre le rang de $\mathcal{K}$ (en tant qu'opérateur) et le nombre de points du processus ponctuel associé.

Proposition 1.0.6 (Théorème 4 de [80], voir aussi [35, 77]). *Soit $\mathcal{K}$ un opérateur intégral satisfaisant Hypothesis 1, et considérons le cas $\alpha = -1$, i.e. $\mu_{\mathcal{K},\lambda}$ est un processus ponctuel déterminantal. On a les propriétés suivantes :*

a) La probabilité que le nombre de points soit fini est respectivement 0 *ou* 1*, et vaut l'un ou l'autre selon que* $\mathrm{Tr}(\mathcal{K})$ *est respectivement fini ou infini. Comme on pouvait s'y attendre, le nombre de points dans un compact $\Lambda \subseteq E$ est fini, puisque $\mathrm{Tr}(\mathcal{K}_\Lambda) < \infty$.*

b) Le hombre de points du processus ponctuel est inférieur ou égal à $n \in \mathbb{N}^*$ avec probabilité 1 *si et seulement si $\mathcal{K}$ est un opérateur de rang fini satisfaisant* $\mathrm{Rang}(\mathcal{K}) \leq n$.

c) Le hombre de points du processus ponctuel est $n \in \mathbb{N}^*$ avec probabilité 1 *si et seulement si $\mathcal{K}$ est un projecteur orthogonal de rang* $\mathrm{Rang}(\mathcal{K}) = n$.

Nous avons aussi besoin d'une condition simple sur les noyaux qui assurent la convergence (faible) des mesures α-déterminantales associées. Ceci est donné par la Proposition 3.10 de [77] :

Proposition 1.0.7. *Soit $(\mathcal{K}^{(n)})_{n\geq 1}$ des opérateurs intégraux positifs avec des noyaux continus $K^{(n)}(x,y)$, $x, y \in E$. Supposons que $\mathcal{K}^{(n)}$ satisfons Hypothesis 1, $n \geq 1$, et que $K^{(n)}$ converge vers un noyau K uniformément sur les compacts lorsque n tend vers l'infini. Alors, le noyau K définit un opérateur intégral $\mathcal{K}$ satisfaisant Hypothesis 3. De plus, pour $\alpha \in \mathbf{M}$, la mesure α-déterminantale $\mu^\alpha_{\mathcal{K}^{(n)},\lambda}$ converge faiblement vers $\mu^\alpha_{\mathcal{K},\lambda}$ quand n tend vers l'infini.*

Enfin, terminons cette section en mentionnant le cas des processus ponctuels determinantaux de projection. On définit pour ce faire un noyau de projection (sur $\{\varphi_n,\ 0 \leq n \leq N\} \subset \mathrm{L}^2(E,\lambda)$) par

$$T_p(x,y) = \sum_{n=0}^{N} \varphi_n(x)\overline{\varphi_n(y)}, \quad \forall x, y \in \mathbb{C} \tag{1.11}$$

où $N \in \mathbb{N}$, et $(\varphi_n)_{n\in\mathbb{N}}$ est une famille orthonormale de $\mathrm{L}^2(E,\lambda)$. On appelle le processus ponctuel α-déterminantal associé un processus ponctuel α-determinantal de projection (sur $\{\varphi_n,\ 0 \leq n \leq N\} \subset \mathrm{L}^2(E,\lambda)$).

Intensité conditionnelle de Papangelou

Propriétés de base

Commençons par définir les mesures de Campbell, et donnons quelques propriétés qu'elles vérifient, telles qu'elles ont été rappelées dans [28]. Le lecteur intéressé pourra trouver plus de détails dans [18, 52].

Definition 1 (Mesures de Campbell). *La mesure de Campbell réduite d'un processus ponctuel μ est la mesure C_μ sur l'espace produit $(E \times \mathcal{X}, \mathcal{B} \otimes \mathcal{F})$ définie par*

$$C_\mu(A \times B) = \int \sum_{x\in\xi} \mathbf{1}_A(x)\mathbf{1}_B(\xi \setminus x)\, \mu(d\xi), \tag{1.12}$$

où $A \in \mathcal{B}$ et $B \in \mathcal{F}$. On définit de manière similaire la mesure de Campbell composée de μ comme la mesure $\hat{C}_\mu$ sur l'espace produit $(\mathcal{X}_0 \times \mathcal{X}, \mathcal{F}_0 \otimes \mathcal{F})$ définie par

$$\hat{C}_\mu(A \times B) = \int \sum_{\alpha\subset\xi,\ \alpha\in\mathcal{X}_0} \mathbf{1}_A(\alpha)\mathbf{1}_B(\xi \setminus \alpha)\, \mu(d\xi),$$

où $A \in \mathcal{F}_0$ et $B \in \mathcal{F}$.

On définit maintenant la condition (Σ_λ) introduite à l'origine dans [63] et [43] comme suit

Hypothesis 2. *On dit que le processus punctual μ satisfait la condition (Σ_λ) si $C_\mu \ll \lambda \otimes \mu$. N'importe quelle densité de Radon-Nikodym c de C_μ par rapport à $\lambda \otimes \mu$ est appelée une version de l'intensité de Papangelou de μ.*

L'hypothèse précédente implique aussi que $\hat{C}_\mu \ll L^\lambda \otimes \mu$ et on noter donc de manière similaire $\hat{c}$ la density de Radon-Nikodym de $\hat{C}_\mu$ par rapport à $L^\lambda \otimes \mu$, et on appelera $\hat{c}$ l'intensité de Papangelou composée de μ. On a alors pour tout $\xi \in \mathcal{X}$, $\hat{c}(\emptyset, \xi) = 1$, de même que pour tout $x \in E$, $\hat{c}(x, \xi) = c(x, \xi)$. L'intensité de Papangelou c de μ est interprétée comme la densité conditionnellement à la configuration $\xi \in \mathcal{X}$. Plus précisément,

$$c(x, \xi)\, \lambda(\mathrm{d}x)$$

est la probabilité de trouver un point dans un voisinage de $x \in E$ conditionnellement à la configuration $\xi \in \mathcal{X}$.
L'intensité de Papangelou composée vérifié la relation de commutation suivante :

$$\forall \eta, \nu \in \mathcal{X}_0, \quad \forall \xi \in \mathcal{X}, \quad \hat{c}(\nu, \eta \cup \xi)\, \hat{c}(\eta, \xi) = \hat{c}(\nu \cup \eta, \xi). \tag{1.13}$$

L'application récursive de la relation précédente donne aussi $\forall x_1, \ldots, x_n \in E, \forall \xi \in \mathcal{X}$,

$$\hat{c}(\{x_1, \ldots, x_n\}, \xi) = \prod_{k=1}^{n} c(x_k, \xi \cup x_1 \cup \cdots \cup x_{k-1}),$$

où on a utilisé la convention $x_0 := \emptyset$.

Hypothesis 4, conjointement avec la définition de la mesure de Campbell réduite, donne lieu à l'identité suivante, connue sous le nom d'identité de Georgii-Nguyen-Zessin :

$$\int_{\mathcal{X}} \sum_{y \in \xi} u(y, \xi \setminus y)\, \mu(\mathrm{d}\xi) = \int_{\mathcal{X}} \int_E u(z, \xi)\, c(z, \xi)\, \lambda(\mathrm{d}z) \mu(\mathrm{d}\xi), \tag{1.14}$$

pour toute fonction positive C_μ-mesurable $u : E \times \mathcal{X} \to \mathbb{R}$. On a aussi une identité similaire pour l'intensité de Papangelou composée :

$$\int_{\mathcal{X}} \sum_{\alpha \subset \xi,\ \alpha \in \mathcal{X}_0} u(\alpha, \xi \setminus \alpha)\, \mu(\mathrm{d}\xi) = \int_{\mathcal{X}} \int_{\mathcal{X}_0} u(\alpha, \xi)\, \hat{c}(\alpha, \xi)\, L^\lambda(\mathrm{d}\alpha) \mu(\mathrm{d}\xi), \tag{1.15}$$

pour toute fonction positive $\hat{C}_\mu$-mesurable $u : \mathcal{X}_0 \times \mathcal{X} \to \mathbb{R}$. En combinant (1.14) et la définition des fonctions de corrélation, on obtient :

$$\mathbb{E}[c(x, \xi)] = \rho_1(x), \tag{1.16}$$

pour presque tout $x \in E$. On obtient aussi de manière plus générale, en utilisant (1.15), que

$$\mathbb{E}[\hat{c}(\alpha, \xi)] = \rho(\alpha), \tag{1.17}$$

pour presque tout $\alpha \in \mathcal{X}_0$.

Intensité de Papangelou des processus ponctuels déterminantaux

Dans cette section, on considère le cas d'un processus punctual déterminantal ($\alpha = -1$) et on rappel un certain nombre de résultats de [28]. Le premier résultat donne l'intensité de Papangelou de μ_Λ, i.e. la restriction de $\mu_{\mathcal{K},\lambda}$ à un compact $\Lambda \subseteq E$.

Theorem 1.0.3 (Théorème 3.1 de [28]). *Pour tout $\Lambda \subseteq E$, μ_Λ satisfait la condition $(\Sigma_{\lambda_\Lambda})$ (ici, λ_Λ est la restriction de λ à un compact Λ). Une version de son intensité de Papangelou composée $\hat{c}_\Lambda$ est donnée par*

$$\hat{c}_\Lambda(\alpha, \xi) = \frac{\det J[\Lambda](\alpha \cup \xi)}{\det J[\Lambda](\xi)}, \quad \alpha \in \mathcal{X}_0,\ \xi \in \mathcal{X}, \tag{1.18}$$

où par convention, le rapport est supposé être nul quand le dénominateur l'est. Cette version de l'intensité de Papangelou composée vérifie en outre

$$\hat{c}_\Lambda(\alpha, \xi) \geq \hat{c}_\Lambda(\alpha, \eta), \qquad \text{and} \qquad 0 \leq \hat{c}_\Lambda(\alpha, \xi) \leq \det J[\Lambda](\alpha) \leq \prod_{x \in \alpha} J[\Lambda](x, x), \tag{1.19}$$

dès que $\xi \subset \eta \in \mathcal{X}_\Lambda$ et $\alpha \subset \mathcal{X}_\Lambda \setminus \eta$.

Définissons maintenant

$$\mathcal{J} := (\mathrm{I} + \alpha\mathcal{K})^{-1}\mathcal{K}, \tag{1.20}$$

qui peut être vu comme un opérateur d'interaction globale. Comme cela a été prouvé dans [28], $\mathcal{J}$ satisfait un certain nombre de bonnes propriétés : c'est un opérateur intégral localement à trace, et son noyau $(x, y) \mapsto J(x, y)$ peut être choisi selon la Proposition 1.0.3. Alors, on a que le processus ponctuel déterminantal est stochastiquement dominé par le processus ponctuel de Poisson avec intensité $x \mapsto J(x, x)\,\lambda(\mathrm{d}x)$, dénoté ici par π_J. La notion de domination stochastique est notée

$$\mu \preceq \pi_J, \tag{1.21}$$

et a été prouvée dans [28]. Voir aussi [29] pour des résultats plus avancés à propos de domination stochastique. Rappelons simplement que pour deux processus ponctuels ν, ν', on dit que ν' domine stochastiquement ν, et on écrit $\nu \preceq \nu'$ si

$$\int f \,\mathrm{d}\nu \leq \int f \,\mathrm{d}\nu',$$

pour toute fonction croissante mesurable f. Ici, on dit que f est croissante si $f(\eta) \leq f(\eta')$ pour $\eta \subset \eta' \in \mathcal{X}$.

Pour nos besoins, il reste à citer le principal théorème de [28] :

Theorem 1.0.4 (Théorème 3.6 de [28]). *Le processus ponctuel déterminantal $\mu_{\mathcal{K},\lambda}$ satisfait la condition (Σ_λ), et son intensité de Papangelou composée est donnée par*

$$\hat{c}(\alpha, \xi) = \lim_{n \to \infty} \hat{c}_{\Delta_n}(\alpha, \xi_{\Delta_n}), \qquad \text{pour } L^\Lambda \otimes \mu \text{ - presque tout } (\alpha, \xi)\ , \tag{1.22}$$

où $(\Delta_n)_{n \in \mathbb{N}}$ est une suite de compacts de E, qui croît vers E.

Notons qu'en général, (1.22) ne donne pas de forme fermée pour les intensités de Papangelou composées. Afin d'avoir une formule fermée pour $\hat{c}$, il est nécessaire d'ajouter des conditions, c.f. Proposition 3.9 de [28]. Plus précisément, définissons

Hypothesis 3. *Supposons $E = \mathbb{R}^d$, λ est la mesure de Lebesgue, et*
- *$\mathcal{J}$ a un noyau intégral continu J.*
- *J a un rayon d'attraction fini $R < \infty$, i.e. $J(x, y) = 0$ if $|x - y| \geq R$.*
- *$\mu_{\mathcal{K},\lambda}$ ne percole pas.*

Sous Hypothesis 3, on a le théorème suivant :

Theorem 1.0.5 (Proposition 3.9 de [28])**.** *Supposons Hypothesis 3. Alors, le processus ponctuel déterminantal $\mu_{\mathcal{K},\lambda}$ satisfait la condition (Σ_λ), et son intensité de Papangelou composée est donnée par*

$$\hat{c}(\alpha, \xi) = \frac{\det J(\xi_W \cup \alpha)}{\det J(\xi_W)} \mathbf{1}_{\operatorname{diam} W(\alpha,\xi) < \infty}, \qquad \text{for } L \otimes \mu_{\mathcal{K},\lambda} \text{ - almost every } (\alpha, \xi) \ , \tag{1.23}$$

où $W(\alpha, \xi)$ est l'union des $B_R(\alpha \cup \xi)$ touchant la configuration α, et $\xi_W := \xi_{W(\alpha,\xi)}$ est la restriction de ξ à $W(\alpha, \xi)$.

Résultats

Nous avons donc introduit les notions nécessaires à l'étude des processus ponctuels, et en particulier des processus ponctuels α-déterminantaux. Nous allons maintenant passer à la présentations des résultats à proprement parler obtenus dans ce manuscrit.

Simulation des processus α-déterminantaux

Nous nous sommes tout d'abord intéressés à des méthodes de simulation pratique des processus ponctuels α-déterminantaux, en même temps que des implémentations numériques de modèles. En effet, la simulation de processus ponctuels déterminantaux était largement ignorée jusqu'à l'article fondateur de [34] dans lequel les auteurs donnent un algorithme pratique pour la simulation de processus ponctuels déterminantaux. Une discussion théorique de l'algorithme ainsi que des résultats statistiques ont aussi été explorés dans [44]. Nous avons eu deux axes d'intérêt principaux dans la thématique de la simulation des processus α-déterminantaux. Nous nous sommes intéressés plus particulièrement au processus ponctuel de Ginibre, pour des raisons que nous expliquerons plus loin. Ce processus ponctuel possède des propriétés qui le rendent pertinent pour la simulation, et nous étudierons donc des manières plus efficaces de le simuler.
Commençons donc à nous intéresser au processus ponctuel de Ginibre, qui a soulevé un grand intérêt depuis son introduction par J. Ginibre dans [30]. Plus récemment, des résultats probabilistes ont été obtenus dans le cas particulier du processus ponctuel de Ginibre, par exemple dans [31,76]. La procédure de simulation qui est sous-entendue dans [30] a été développée complètement d'un point de vue numérique dans [12]. La première utilisation du processus ponctuel de Ginibre en tant que modèle semble remonter à [12]. Plus récemment, dans [54,84,87], différents auteurs ont utilisé le processus ponctuel de Ginibre pour modéliser des phénomènes apparaissant en réseaux. En effet, ce modèle a de nombreux avantages en vue d'applications : il est invariant par rapport aux rotations et aux translations, ce qui lui donne un compact naturel sur lequel le simuler : la boule centrée autour de l'origine. De plus, la répulsion électrostatique entre les particules semble être réalise et modeler de nombreuses applications. Notre objectif dans ce papier est d'étudier la simulation du processus ponctuel de Ginibre d'un point de vue pratique, et donner différentes

méthodes qui seront plus ou moins appropriées selon l'application qui nous concernera. Le problème principal qu'il nous faudra contourner est que, même si les valeurs propres de matrices de l'ensemble GUE forment un processus ponctuel de Ginibre, ces valeurs propres ne sont pas distribuées sur un compact, même si nous remarquerons que quand la taille de la matrice N tend vers l'infini, elles tendent à l'être (résultat connu sous le nom de loi circulaire en théorie des matrices aléatoires). De plus, comme nous le verrons plus loin, tronquer le processus ponctuel à un compact naturel et faire tendre N vers l'infini n'est pas la manière optimale de procéder, quoique cette opération conserve la nature déterminantale du processus ponctuel. En fait, nos méthodes se baseront sur une modification du noyau associé classiquement au processus ponctuel de Ginibre. Nous étudierons en détail la projection de ce noyau sur un compact, sa troncature à un rang fini, et enfin une combinaison de ces deux opérations. Chacune de ces opérations sur le noyau aura des résultats différents sur le processus ponctuel résultant, ainsi que sur les techniques de simulations optimales associées.

Pour être plus précis concernant nos résultats, introduisons ce que nous appellerons le processus ponctuel de Ginibre. Le processus ponctuel de Ginibre, noté μ_{Gin} dans le reste de ce résumé est défini comme le processus ponctuel de Ginibre sur $\mathbb{C}$ avec comme noyau

$$K_{\mathrm{Gin}}(z_1, z_2) = \frac{1}{\pi} e^{z_1 \bar{z_2}} e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}, \quad z_1, z_2 \in \mathbb{C}, \tag{1.24}$$

et par rapport à la mesure sous-jacente $\lambda := \mathrm{d}\ell(z)$, la mesure de Lebesgue sur $\mathbb{C}$ (i.e. $\mathrm{d}\ell(z) = \mathrm{d}x\,\mathrm{d}y$, lorsque $z = x + iy$). Le noyau K_{Gin} peut être décomposé de manière naturelle de la manière suivante :

$$K_{\mathrm{Gin}}(z_1, z_2) = \sum_{n\geq 0} \phi_n(z_1)\overline{\phi_n(z_2)}, \quad z_1, z_2 \in \mathbb{C},$$

où $\phi_n(z) := \frac{1}{\sqrt{\pi n!}} e^{-\frac{1}{2}|z|^2} z^n$, pour $n \in \mathbb{N}$ et $z \in \mathbb{C}$. On vérifiera facilement que $(\phi_n)_{n\in\mathbb{N}}$ est une famille orthonormale de $\mathrm{L}^2(\mathbb{C}, \mathrm{d}\ell)$. En fait, $(\phi_n)_{n\in\mathbb{N}}$ est un sous-ensemble dense de $\mathrm{L}^2(\mathbb{C}, \mathrm{d}\ell)$. Le processus ponctuel de Ginibre μ_{Gin} vérifie les propriétés basiques suivantes :

Proposition 1.0.8. *Le processus de Ginibre μ_{Gin}, i.e. le processus ponctuel déterminantal ayant pour noyau K_{Gin}, satisfait :*

- *μ_{Gin} est ergodique par rapport aux translations sur le $\mathbb{R}^2$.*
- *μ_{Gin} est isotropique.*
- *$\mu_{\mathrm{Gin}}(\mathbb{C}) = +\infty$ presque sûrement, i.e. le processus ponctuel de Ginibre a une infinité de points presque sûrement.*

Puisque μ_{Gin} a une infinité de points presque sûrement, il n'est pas possible de le simuler directement. Ainsi, nous chercherons dans la suite à modifier le noyau K_{Gin} de façon à obtenir certaines versions du processus ponctuel de Ginibre qui seront elles possibles à simuler.

La première idée est de considérer le processus de Ginibre tronqué, défini pour $N \in \mathbb{N}_*$ par

$$K^N_{\mathrm{Gin}}(z_1, z_2) = \sum_{n=0}^{N-1} \phi_n(z_1)\overline{\phi_n(z_2)}, \quad z_1, z_2 \in \mathbb{C}, \tag{1.25}$$

et qui est en fait la troncature de (1.24) au n-eme terme. De plus, on note μ^N_{Gin} le processus ponctuel déterminantal associé, aver comme mesure sous-jacente $\mathrm{d}\ell$. On remarque

que μ_{Gin}^N tend vers μ_{Gin} faiblement, quand N tend vers l'infini. Comme c'est un noyau de projection du type (1.11), nous avons vu précédemment que μ_{Gin}^N avait N points presque sûrement. μ_{Gin}^N n'est par contre clairement plus invariant par translations ; par contre, il reste isotropique pour les mêmes raisons que μ_{Gin} l'est. Physiquement, μ_{Gin}^N représente la distribution de N électrons polarisés dans un champ magnétique perpendiculaire, remplissant les N niveaux de Landau les plus bas, ainsi que remarqué dans [74].

La simulation du processus déterminantal avec pour noyau (1.25) est relativement simple. En effet, puisque le processus ponctuel a N points presque sûrement, il ne reste qu'à simuler l'emplacement des points. Pour ce cas particulier, il n'y a même pas besoin de tirer les points selon la densité jointe puisqu'il a été prouvé dans [30] que les valeurs propres d'une matrice hermitienne $N \times N$ avec des entrées gaussiennes sont distribuées selonμ_{Gin}^N. Plus précisément, considérons une matrice $A := (A_{nm})_{1\leq n,m\leq N}$, telle que pour $1 \leq n, m \leq N$,

$$A_{nm} = \frac{1}{\sqrt{2}} \left(A_{nm}^1 + iA_{nm}^2\right),$$

où $A_{nm}^1, A_{nm}^2 \sim \mathcal{N}(0,1)$, $1 \leq n, m \leq N$ sont des gaussiennes indépendantes. Alors, les valeurs propres de A ont pour distribution μ_{Gin}^N. Cette manière de simuler est de loin la plus efficace dans ce cas.
Cette méthode de simulation, quoique efficace, possède un désavantage de taille : le support de la loi du processus ponctuel déterminantal associé a pour support tout $\mathbb{C}$. De plus, le fait de projeter le déterminantal sur un compact rend le nombre de points du processus ponctuel aléatoire. Ainsi, ce premier modèle sera utile principalement dans les applications pour lesquelles il n'est pas nécessaire que le processus ponctuel soit dans un compact de $\mathbb{C}$ fixé à l'avance.

Après un certain nombre de considérations liées au compact sur lequel nous comptons simuler, nous sommes en mesure d'introduire un nouveau noyau, donné par

$$\tilde{K}_{\text{Gin}}^N(z_1, z_2) = \sum_{n=0}^{N-1} \phi_n^N(z_1)\overline{\phi_n^N(z_2)}, \quad z_1, z_2 \in \mathcal{B}_R, \tag{1.26}$$

et où ϕ_n^N correspond à la fonction ϕ_n restreinte au compact $\mathcal{B}_{\sqrt{N}}$ (après renormalisation). Commençons par prouver que $\tilde{\mu}_{\text{Gin}}^N$, le processus ponctuel déterminantal avec noyau $\tilde{K}_{\text{Gin}}^N$, converge vers μ_{Gin} faiblement quand N tend vers l'infini. C'est une conséquence de Proposition 1.0.7, et nous l'avons prouvé dans la proposition suivante :

Theorem 1.0.6. *$\tilde{K}_{\text{Gin}}^N$ converge uniformément sur les compacts vers K_{Gin} quand N tend vers l'infini. En conséquence, les mesures déterminantes correspondantes convergent faiblement vers le processus ponctuel déterminantal de noyau K_{Gin}.*

Revenons maintenant au problème de la simulation d'un processus ponctuel déterminantal de noyau (1.26). En tant que processus ponctuel déterminantal de projection, il est simulé de manière efficace par l'algorithme de [34]. Par contre, notons que l'étape de génération des variables aléatoires de Bernoulli n'est plus nécessaire, puisque ce processus ponctuel possède N points presque sûrement. Pour résumer, la simulation du processus ponctuel associé au noyau (1.26) sur une boule centrée autour de l'origine et de rayon $a \geq 0$ se fera selon l'algorithme suivant :

Algorithm 1 Simulation du processus de Ginibre tronqué sur un compact

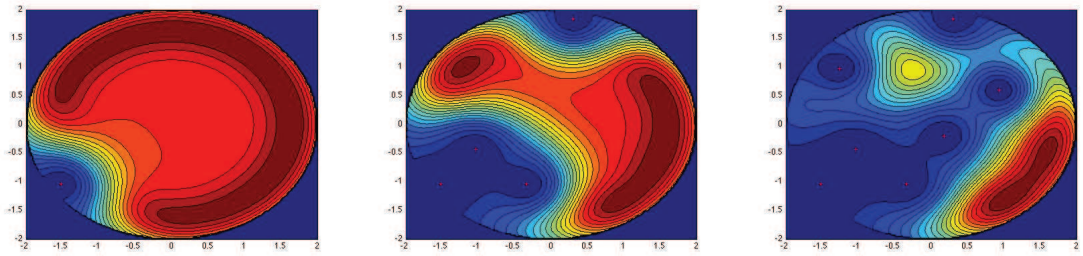
définir $\phi_k(z) = \frac{N}{\pi a^2 \gamma(k+1,N)} e^{-\frac{N}{2a^2}|z|^2} (\frac{Nz}{a^2})^k$, pour $z \in \mathcal{B}_N$ et $0 \leq k \leq N-1$.
définir $\mathbf{v}(z) := (\phi_0(z), \ldots, \phi_{N-1}(z))$, pour $z \in \mathcal{B}_N$.
tirer X_N à partir de la loi de densité $p_N(x) = \|\mathbf{v}(x)\|^2/N, \ \ x \in \Lambda$
poser $\mathbf{e}_1 = \mathbf{v}(X_N)/\|\mathbf{v}(X_N)\|$
for $i = N-1 \to 1$ **do**
  tirer X_i à partir de la loi de densité

$$p_i(x) = \frac{1}{i}\Big[\|\mathbf{v}(x)\|^2 - \sum_{j=1}^{N-i} |\mathbf{e}_j^* \mathbf{v}(x)|^2\Big]$$

  set $\mathbf{w}_i = \mathbf{v}(X_i) - \sum_{j=1}^{N-i} \Big(\mathbf{e}_j^* \mathbf{v}(X_i)\Big) \mathbf{e}_j, \quad \mathbf{e}_{N-i+1} = \mathbf{w}_i/\|\mathbf{w}_i\|$
end for
return $(X_1, \ldots, X_N)$

Le processus ponctuel résultant est donc un processus ponctuel de noyau (1.26). Son support est $\mathcal{B}_a$ et ce processus a N points presque sûrement.

Nous donnons enfin le résultat de l'application de cet algorithme et l'obtention d'un tirage du processus ponctuel associé.



Ce processus ponctuel déterminantal a l'avantage d'être facile à utiliser dans les simulations, et d'avoir un nombre de points fixé à l'avance. De plus, Théorème 4.4.1 prouve sa convergence vers le processus ponctuel de Ginibre quand N tend vers l'infini.

Analyse stochastique et intégration par parties

Afin de présenter nos résultats en analyse stochastique, nous procédons comme dans [2] et introduisons le même gradient ainsi que les mêmes fonctions test. Plus précisément, on dit que $F : \mathcal{X} \to \mathbb{R}$ est dans $\mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X})$ si F est de la forme

$$F : \xi \longmapsto f(\sum_{x \in \xi} h_1(x), \ldots, \sum_{x \in \xi} h_N(x)),$$

pour un certain $N \in \mathbb{N}$, $h_1, \dots, h_N \in \mathcal{D} = C^\infty(E)$, $F \in C_b^\infty(\mathbb{R}^N)$. Pour une fonction F dans $\mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})$, on définit son gradient comme

$$\nabla F(\xi) = \sum_{i=1}^{N} \partial_i f(\sum_{x\in\xi} h_1(x), \dots, \sum_{x\in\xi} h_N(x)) \sum_{x\in\xi} \nabla^E h_i(x),$$

où ∇^E est le gradient usuel sur E. Rappelons maintenant un certain nombre de résultats de [2] obtenus dans le cas du processus ponctuel de Poisson. Il a été montré qu'il existe une formule d'intégration par parties sur $\mathrm{L}^2(\mathcal{X}, \mu)$ au sens où il existe un opérateur, que nous appellerons divergence, et noterons div, et tel qu'on ait la formule suivante :

$$\mathbb{E}[F\,\nabla G] = \mathbb{E}[G\,\mathrm{div} F],$$

pour $F, G \in \mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})$. Puis, après l'étude du générateur $\mathcal{H}F = \mathrm{div}\nabla F$, il y est montré qu'il existe une diffusion quasi-continue correctement associée avec le processus ponctuel de Poisson. Dans ce cas, la diffusion est un mouvement brownien sur l'espace des configurations. La même étude a été effectuée dans [3] pour le processus ponctuel de Gibbs, et cette étude a été initiée dans [14] pour le cas d'un processus ponctuel déterminantal, et il y a été obtenu une formule d'intégration par parties. Notre objectif dans une première étape est de clarifier un certain nombre d'arguments de [14] qui n'étaient pas suffisamment rigoureux. Puis, nous effectuerons la suite du cheminement effectué dans [2, 3] dans le cas d'un processus ponctuel déterminantal, i.e. nous construisons la diffusion associée avec le processus ponctuel déterminantal. Nous allons maintenant expliquer plus précisément cette dernière étape

Nous construisons des formes de Dirichlet associées aux processus ponctuels déterminantaux, et nous les utilisons pour prouver l'existence de diffusions correctement associées aux processus ponctuels déterminantaux. À cette fin, nous donnons une formule d'intégration par partie pour des fonctions du processus ponctuel qui est basée sur un résultat de quasi-invariance prouvé dans [14]. Puis, cette formule d'intégration par parties est étendue à la fermeture des opérateurs de gradient et de divergence. Notre approche est similaire à celle de [2] et notre construction diffère de celle de [83] qui est basée sur des identités trajectorielles. Une telle construction peut être utilisée pour obtenir des formules d'estimation de densités et d'analyse de sensibilité pour des fonctions des processus ponctuels déterminantaux, du type celles de [69].

Notre théorème principal donne la forme de Dirichlet symétrique associée à un processus ponctuel déterminantal :

Theorem 1.0.7 (Intégration par parties sur les compacts pour des processus ponctuels déterminantaux)**.** *Supposons que $\mathcal{K}, \lambda$ satisfons Hypothesis 1, ainsi que des conditions d'intégrabilité suffisantes. Pour F et G deux fonctions cylindriques de $\mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda)$, et tout compact $\Lambda \subset E$, on a*

$$\begin{aligned}\int_{\mathcal{X}_\Lambda} \nabla_v F(\xi)G(\xi)\,\mu_{K_\Lambda,\lambda}(d\xi) = &-\int_{\mathcal{X}_\Lambda} F(\xi)\nabla_v G(\xi)\,\mu_{K_\Lambda,\lambda}(d\xi)\\ &+\int_{\mathcal{X}_\Lambda} F(\xi)G(\xi)(B_v^\lambda(\xi) + \nabla_v U[\Lambda](\xi))\,\mu_{K_\Lambda,\lambda}(d\xi), \quad (1.27)\end{aligned}$$

où B_v^λ sera défini plus tard en (5.11).

Une application aux dynamiques stochastiques associées est ensuite obtenue, et nous caractérisons les diffusions associées au processus ponctuel déterminantal.
Tous les résultats de cette sous-section seront en fait généralisés à un processus ponctuel quelconque, ou presque. Plus précisément, nous considérons un processus ponctuel qui admet des intensités conditionnelles de Papangelou. Alors, sous des conditions d'intégrabilité, nous prouvons comme dans le cas déterminantal l'existence de diffusions correctement associées au processus ponctuel. Ceci nous permet de généraliser au passage les résultats de [14], au sens où nos résultats incluent le cas d'un processus ponctuel déterminantal possédant une intensité de Papangelou, i.e. tous les exemples de [28].

Formules de moments pour des processus ponctuels généraux

Notre objectif ici est de généraliser les formules de [66,67] au cas d'un processus ponctuel quelconque, ou presque. Pour obtenir nos résultats, nous ne pouvons recourir aux mêmes types de preuves que dans [68], vu que celles-ci reposent principalement sur des outils de calcul de Malliavin développés uniquement dans le cadre poissonnien. Pour être plus précis, nous cherchons à calculer les moments de tous ordres d'objets du type $\sum_{x\in\xi} u(x,\xi)$, où ξ est une configuration sur un espace Polonais et u un processus stochastique. Ainsi, nous obtenons le Théorème 1.0.8 qui généralise ce qui a été obtenu dans [68] pour un processus ponctuel de Poisson. Ce théorème généralise aussi ceux de [17] qui concernent un processus ponctuel général, mais uniquement pour une fonction $u(x,\xi)$ ne dépendant que de x. Notre démonstration est basée principalement sur l'équation de Georgii-Nguyen-Zessin rappelée en (1.14). Elle implique la notion d'intensité de Papangelou $c(x,\xi)$, définie intuitivement comme la densité de probabilité d'avoir une particule en x sachant que l'on observe la configuration ξ.
Puis, nous en déduisons un opérateur de divergence δ, traditionnellement défini pour un processus ponctuel de Poisson, et que nous généralisons à un processus ponctuel quelconque via la formule (1.29). Nous sommes de même en mesure de calculer les moments de $\delta(u)$, mais surtout en déduisons le Corollaire 1.0.3 qui généralise la formule d'isométrie bien connue dans le cas du processus ponctuel de Poisson.
Enfin, nous considérons une transformation $\tau(x,\xi)$ qui perturbe les particules x d'une configuration ξ selon $\tau(\xi) := \sum_{x\in\xi}\delta_{\tau(x,\xi)}$, où ici δ est la fonction de Dirac. Nous caractérisons dans le Théorème 1.0.9 la mesure transformée. Les conditions imposées ici sur la transformation τ sont les mêmes que dans le cas du processus ponctuel de Poisson.

Theorem 1.0.8. *Pour tout $n\in\mathbb{N}$, toutes fonctions mesurables $u_k : E\times\mathcal{X}\to\mathbb{R}$, $k=1,\dots,n$, et toute fonction mesurable positive F sur $\mathcal{X}$, on a*

$$\mathbb{E}[F\prod_{k=1}^{n}\int u_k(y,\xi)\,\xi(dy)] = \sum_{k=1}^{n}\sum_{\mathcal{P}\in\mathcal{T}_n^k}\mathbb{E}\Big[\int_{E^k} F(\xi\cup x)u^{\mathcal{P}}(x,\xi\cup x)c(x,\xi)\,\lambda_k(dx)\Big],$$

où $\mathcal{T}_n^k$ est l'ensemble des partitions de $\{1,\dots,n\}$ en k sous-ensembles. Ici, pour $\mathcal{P}=\{P_1,\dots,P_k\}\in\mathcal{T}_n^k$, on utilise la notation plus compacte $x := (x_1,\dots,x_k)$, ainsi que $\lambda_k(dx) := \lambda(dx_1)\dots\lambda(dx_k)$ et

$$u^{\mathcal{P}}(x,\xi) := \prod_{l=1}^{k}\prod_{i\in P_l} u_i(x_l,\xi).$$

Cette formule généralise ainsi un certain nombre de formules obtenues dans [66], puisqu'elles sont ici valables pour un processus ponctuel quelconque. En particulier, nous obtenons

Corollary 1.0.1. *Pour tout $n \in \mathbb{N}$, et toute fonction mesurable v sur E, on a*

$$\mathbb{E}[F\big(\int v(y)\,\xi(dy)\big)^n] = \sum_{k=1}^{n} \sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_n^k} \int_{E^k} v(x_1)^{|P_1|}\dots v(x_k)^{|P_k|} \\ \mathbb{E}[F(\xi\cup x_1\cup\dots\cup x_k)c(\{x_1,\dots,x_k\},\xi)]\,\lambda(dx_1)\dots\lambda(dx_k).$$

Avant d'aller plus loin, nous introduisons une hypothèse supplémentaire.

Hypothesis 1. *Supposons que pour tout i et tout entier k,*

$$\mathbb{E}[\xi(\Lambda)^k\big(\int_\Lambda c(z,\xi)\,\lambda(dz)\big)^i] < \infty,$$

pour tous compacts $\Lambda \subset E$.

Cette hypothèse est en particulier vérifiée pour des processus α-déterminantaux (voir [28, 77]). Alors, nous sommes en mesure de prouver :

Proposition 1.0.9. *Qupposons que Hypothesis 1 est vérifiée. Alors, pour tout $n \in \mathbb{N}$, tout processus borné $u : E\times\mathcal{X}\to\mathbb{R}$ avec un support compact sur E, et toute fonction bornée F sur $\mathcal{X}$, on a*

$$\mathbb{E}[F\big(\int u(y,\xi)\,\nu(dy)\big)^n] = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{k=1}^{n-i}\sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_{n-i}^k}$$

$$\mathbb{E}\Big[\int_{E^k} c(x,\xi)F(\xi\cup x)\Big(\int u(z,\xi\cup x)c(z,\xi\cup x)\,\lambda(dz)\Big)^i\prod_{l=1}^{k}u(x_l,\xi\cup x)^{|P_l|}\,\lambda_k(dx)\Big]. \tag{1.28}$$

Dans le contexte du calcul de Malliavin appliqué aux processus de Poisson, nous savons que l'opérateur différence admet comme adjoint l'intégrale stochastique compensée. Les formules précédentes impliquent une généralisation élégante de la formule d'intégration par parties sur l'espace de Poisson à un processus ponctuel quelconque. De manière équivalente, nous obtenons une formule de caractérisation de l'adjoint de l'opérateur différence. Commençons par définir ces différentes notions.

Definition 2 (Opérateur différence)**.** *Pour $F : \mathcal{X}\to\mathbb{R}$, on définit DF, l'opérateur différence appliqué à F, comme suit :*

$$\begin{array}{rccc} DF: & E\times\mathcal{X} & \longrightarrow & \mathbb{R} \\ & (x,\xi) & \longmapsto & D_xF(\xi) = F(\xi\cup x) - F(\xi\setminus x). \end{array}$$

Definition 3 (Opérateur divergence)**.** *On dit qu'une fonction mesurable $u : E\times\mathcal{X}\to\mathbb{R}$ appartient à $\mathrm{Dom}(\delta)$ dès lors que $\mathbb{E}[\int |u(y,\xi)|\,c(y,\xi)\,\lambda(dy)]$ est finie. Alors, pour $u\in\mathrm{Dom}(\delta)$, on définit $\delta(u)$ par*

$$\delta(u) = \int u(y,\xi\setminus y)\,\xi(dy) - \int u(y,\xi)\,c(y,\xi)\,\lambda(dy). \tag{1.29}$$

Il s'ensuit par Proposition 1.0.9 la formule suivante :

Corollary 1.0.2 (Formule de dualité)**.** *Pour toute fonction bornée F sur $\mathcal{X}$, et tout processus $u : (x, \xi) \mapsto u(x, \xi)$, C_μ-mesurable, et dans* $\mathrm{Dom}(\delta)$*, on a*

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\int_E D_z F(\xi) u(z, \xi) c(z, \xi)\, \lambda(dz)]. \tag{1.30}$$

Comme corollaire, nous obtenons aussi une formule d'isométrie modifiée :

Corollary 1.0.3. *Pour un processus $u : (x, \xi) \mapsto u(x, \xi)$, C_μ-mesurable, et dans* $\mathrm{Dom}(\delta)$*, on a*

$$\begin{aligned}\mathbb{E}[\delta(u)^2] = \mathbb{E}[\int u(y,\omega)^2 c(y,\omega) d\lambda(y)] + \mathbb{E}[\iint D_y u(z,\,\omega) D_z u(y,\,\omega) c(\{y,\,z\},\,\omega) d\lambda(y) d\lambda(z)] \\ - \mathbb{E}[\iint u(z,\,\omega) u(y,\,\omega) \Big(c(\{y,\,z\},\,\omega) - c(y,\,\omega) c(z,\,\omega) \Big) d\lambda(y) d\lambda(z)].\end{aligned}$$

Nous pouvons noter que les deux premiers termes apparaissent dans la formule correspondante pour le processus ponctuel de Poisson. Par contre, le dernier terme est intrinsèquement lié à la structure de corrélation du processus ponctuel.

Enfin, donnons une application des formules de moments obtenues précédemment. Nous avons pour objectif d'étudier une transformation aléatoire du processus ponctuel général μ. Dans la suite, nous nous intéresserons à la condition suivante :

Hypothesis 2. *Pour $u : E \times \mathcal{X} \to \mathbb{R}$, on suppose que u vérifie*

$$D_{x_1} \dots D_{x_k} u(x_1, \xi) \dots u(x_k, \xi) = 0, \qquad \text{for } C_\mu\text{-a.e. } (x, \xi) \in E \times \mathcal{X}$$

pour tout $k \geq 1$.

Maintenant, considérons la transformation aléatoire $\tau : E \times \mathcal{X} \to E$. Pour $\xi \in \mathcal{X}$, considérons la mesure image de ξ par τ, que nous appellerons la transformation aléatoire $\tau_*(\xi)$, définie par

$$\tau_*(\xi) = \sum_{x \in \xi} \delta_{\tau(x,\xi)},$$

et donc τ_* déplace chacun des points de la configuration dans la direction τ. Nous cherchons à étudier l'effet de cette transformation sur la mesure μ sous des hypothèses suffisamment fortes sur τ. Spécifiquement, nous introduisons la condition suivante :

(H1) La transformation aléatoire τ_* satisfait Hypothesis 12, au sens que pour tout $u : \mathcal{X} \to \mathbb{R}$, $u \circ \tau_*$ vérifie Hypothesis 12.

(H2) Pour presque tout $\xi \in \mathcal{X}$, $\tau(\cdot, \xi)$ est inversible, et nous noterons son inverse $\tau^{-1}(x, \xi)$, $x \in E, \xi \in \mathcal{X}$. On noter aussi $\tau_*^{-1}(\xi)$ la mesure image de ξ par τ^{-1}.

Nous sommes maintenant en mesure de donner le résultat principal :

Theorem 1.0.9. *Soit $\tau : E \times \mathcal{X} \to E$ une transformation aléatoire ainsi que définie précédemment, et satisfaisant* (**H1**) *et* (**H2**)*. Supposons que τ transforme λ en σ, i.e. $\tau(\cdot, \xi)\lambda = \sigma$, $\xi \in \mathcal{X}$, où σ est une mesure fixée sur $(E, \mathcal{B})$. Alors, $\tau_*\mu$ a des functions de corrélation par rapport à σ qui sont données par*

$$\rho_\tau(x_1, \dots, x_k) = \mathbb{E}\big[c(\{\tau^{-1}(x_1, \xi), \dots, \tau^{-1}(x_k, \xi)\}, \xi)\big], \quad x_1, \dots, x_k \in E. \tag{1.31}$$

Ce théorème généralise tous les résultats connus. Remarquons en particulier que dans le cas d'un processus ponctuel de Poisson, le théorème précédent s'écrit de la manière suivante.

Corollary 1.0.4. *Soit $\mu = \pi^{d\lambda}$ un processus de Poisson avec intensité λ. Soit $\tau : E \times \mathcal{X} \to E$ une transformation aléatoire satisfaisant* (**H1**) *et* (**H2**)*. Supposons de plus que τ transforme λ en σ, i.e. $\tau(\cdot, \xi)\lambda = \sigma$, $\xi \in \mathcal{X}$. Alors, τ transforme $\pi^{d\lambda}$ en $\pi^{d\sigma}$.*

Ces résultats ont donné lieu aux publications suivantes :

Publications

— Laurent Decreusefond, Ian Flint, Nicolas Privault, Giovanni Luca Torrisi. *Determinantal processes : a survey.*
À paraître dans "Stochastic analysis for Poisson point processes : Malliavin calculus, Wiener-Itô chaos expansions and stochastic geometry", G. Peccati and M. Reitzner editors, Bocconi & Springer Series, 2014.

— Laurent Decreusefond, Ian Flint. *Moment formulae for general point processes.* Accepté dans "Comptes rendus Mathématique", 2013.

Preprints

— Laurent Decreusefond, Ian Flint, Anaïs Vergne. *Efficient simulation of the Ginibre point process.* Soumis, 2013.
http ://hal.archives-ouvertes.fr/hal-00869259
Inclus dans Chapitre 4.

— Laurent Decreusefond, Ian Flint. *Moment formulae for general point processes.* Soumis à JFA, 2013.
http ://hal.archives-ouvertes.fr/hal-00753801
Inclus dans Chapitre 6.

— Anaïs Vergne, Ian Flint, Laurent Decreusefond and Philippe Martins. *Homology based algorithm for disaster recovery in wireless networks.* Soumis, 2013.
http ://hal.archives-ouvertes.fr/hal-00800520
Inclus dans Appendice A.

— Laurent Decreusefond, Ian Flint, Nicolas Privault, Giovanni Luca Torrisi. *Stochastic dynamics of determinantal processes by integration by parts.* Soumis, 2013.
http ://arxiv.org/abs/1210.6109
Inclus dans Appendice B, voir aussi Section 5.3

— Laurent Decreusefond, Ian Flint, Kah Choon Low. *Perfect Simulation of Determinantal Point Processes.* Soumis, 2013.
http ://hal.archives-ouvertes.fr/hal-00879101
Inclus dans Appendice C.

Chapter 2

Introduction

Background

In this manuscript, we consider a Polish space E, which in examples will be taken to be $E = \mathbb{R}$ or $E = \mathbb{R}^d$. E is said to be the underlying space, and is endowed with a diffusive Radon measure λ, which we call the intensity measure. We call configuration space, and denote by $\mathcal{X}$, the space of subsets of E which have a finite number of points in compact sets. It is then natural to topologize $\mathcal{X}$ with the vague topology, and to define $\mathcal{F}$ the corresponding Borel σ-algebra. Then, we call random point process a triplet $(\mathcal{X}, \mathcal{F}, \mu)$, where μ is a probability measure on $(\mathcal{X}, \mathcal{F})$. Intuitively, a random point process is thus a random variable with state space $\mathcal{X}$.
In the following, we write x for an element of E, while we favor ξ to denote elements of the configuration space $\mathcal{X}$. ξ also sometimes denotes the canonical random variable on $(\mathcal{X}, \mathcal{F}, \mu)$, i.e. ξ is sometimes treated as the random variable with distribution μ.
A random point process μ is characterized by a wide variety of functionals which give information on some of its properties. Let us recall some of them here. First, the so-called correlation functions ρ_n, indexed by $n \in \mathbb{N}$, which are defined in such a way that

$$\rho_n(x_1, \dots, x_n)\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n)$$

is the probability of finding a particle in the vicinity of each x_i, $i = 1, \dots, n$. Therefore, correlation functions indicate repulsiveness or attractiveness since

$$\rho_2(x, y) \leq \rho_1(x)\rho_1(y), \quad x, y \in E \tag{2.1}$$

signals repulsiveness, while

$$\rho_2(x, y) \geq \rho_1(x)\rho_1(y), \quad x, y \in E$$

signals attractiveness. The other tool of major importance which will be useful throughout the manuscript is the so-called Papangelou (conditional) intensity. Intuitively, the Papangelou intensity c is such that $c(x, \xi)\, \lambda(\mathrm{d}x)$ is the conditional probability of finding a particle in the vicinity of x, given the configuration ξ. Historically, the first type of processes satisfying this condition is the Gibbs process. For a Gibbs process, $c(x, \xi) = e^{H(x \cup \xi) - H(\xi)}$, where H is a global energy function, chosen in a suitable class of functions.

Random point processes have been extensively studied. However, more often than not, the measure μ is the Poisson measure, in which case extensive work has been done. However the Poisson measure has an important flaw, which is that the points do not have a

correlation structure. To visualize this, recall how a Poisson process is simulated. First, we are to simulate the number of points in a certain compact set, and then we simulate each point uniformly on the compact set, independently of the other points. As such, the Poisson measure rarely fits realistic data, which often exhibits correlation between events. The question is then of the modeling of such correlation, while maintaining a satisfying mathematical framework.
A random point process being a priori very general, we have very few constraints when choosing a model. One example of such a model traces back to [58], and is known as the Gibbs point process. This point process of major interest in physics models simple interactions between particles, given in the form of a potential. In a certain sense, the global interaction of a system of particles on a certain particle can be modeled by a sum of one to one interactions. There is a satisfying framework for Gibbsian measures, but some smoothness conditions on the potential are quite restrictive. Gibbs point processes model one-to-one interactions which appear in physics quite well, and can be generalized to more general k-body interactions.

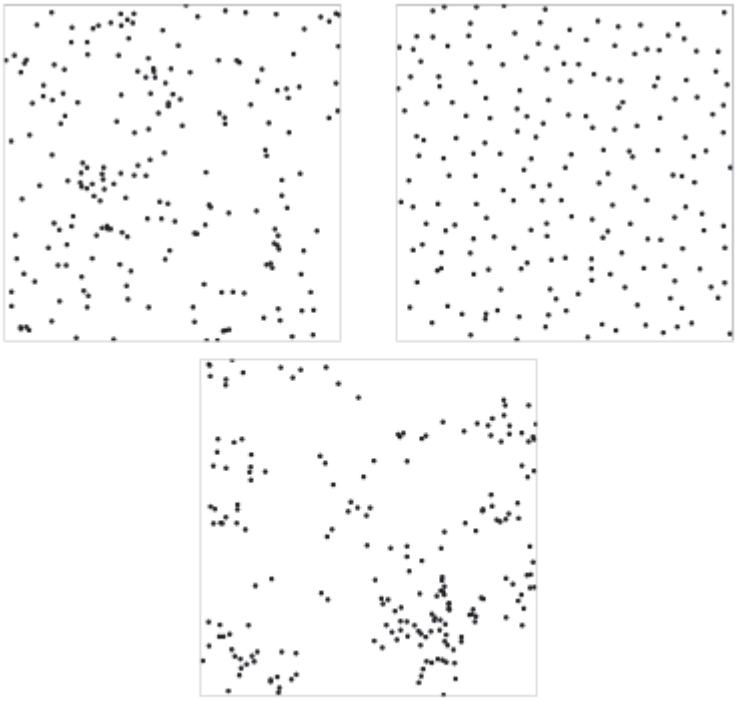


Figure 2.1: Comparison [2] of Poisson point process (top left), determinantal point process (top right), and permanental point process (bottom)

2. Reference for the image: [34]

Determinantal processes were introduced by O. Macchi in 1975 as a mathematical model which model the positioning of fermion particles. The analog - the permanental process, which is associated with bosons - was shown to be linked to determinantal processes by T. Shirai and Y. Takahashi, both types of processes being incorporated in a common class. In some sense, these two types of processes are extreme cases of a general type of point process, that have a particular correlation structure. In this manuscript, we call these processes α-permanental point processes. These point processes were studied in depth in [77], which is one of our main references for the general framework concerning these point processes.
Mathematically speaking, α-determinantal point processes are defined via their correlation functions. To be more precise, we define for a square matrix $A = (A_{ij})_{1\leq i,j\leq n}$ of size $n\times n$, the α-determinant $\det_\alpha A$ by:

$$\det_\alpha A = \sum_{\sigma\in S_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} A_{i\sigma(i)},$$

where S_n stands for the n-th symmetric group and $\nu(\sigma)$ is the number of cycles in the permutation $\sigma \in S_n$. Then, an α-determinantal point process is chosen so as to have its correlation functions satisfy

$$\rho_k(x_1,\dots,x_k) = \det_\alpha(K(x_i,x_j))_{1\leq i,j\leq k},$$

for any $k \geq 1$ and $x_1,\dots,x_k \in E$. Here K is a quite general operator which will satisfy some hypotheses defined later on. Most importantly, K is chosen to be symmetric and with spectrum in $[0,1]$. We notice straight away that when $\alpha = -1$, we obtain the usual determinant and hence (2.1) is trivially satisfied by symmetry of K.
Although these general α-permanental point processes were grouped together in [77], the correlation structure of these point processes is very different depending on the sign of α, which leads us to categorize the resulting process as one modeling either fermion or boson particles. To be more precise, when $\alpha > 0$, the point process exhibits attraction, and is used to model bosons. On the contrary, when $\alpha < 0$, the point process exhibits repulsion and is used to model fermions. This profound difference in the correlation structure lead us to focus our attention in this thesis to the case of $\alpha < 0$. Although some of our results could be stated in a very general setting and incorporate the $\alpha \geq 0$ case, we use abundantly the fact that the point process is repulsive, specifically to obtain a stochastic domination by the well-known Poisson process, which is known not to be true when $\alpha \geq 0$, see [23].
The difference in the correlation properties of these different processes is shown in Figure 2.2. Let us also point out that α serves as a parameter which gives rise to an intermediate class of point processes between the extreme cases that are determinantal and permanental point processes. We also give in Figure 2.3 a sample of Dyson's model, which is a determinantal process on $E := \mathbb{R}$ with the sine kernel. Dyson's model traces back to [22], see also [81] for further developments.

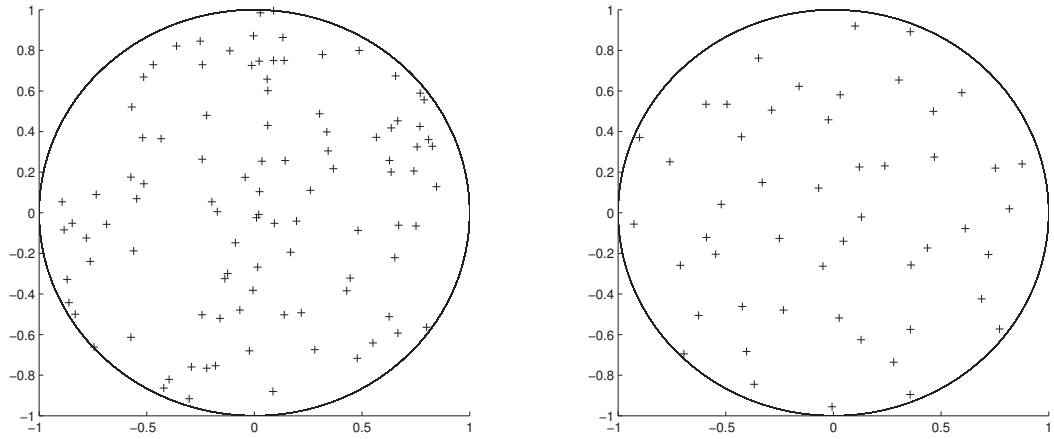
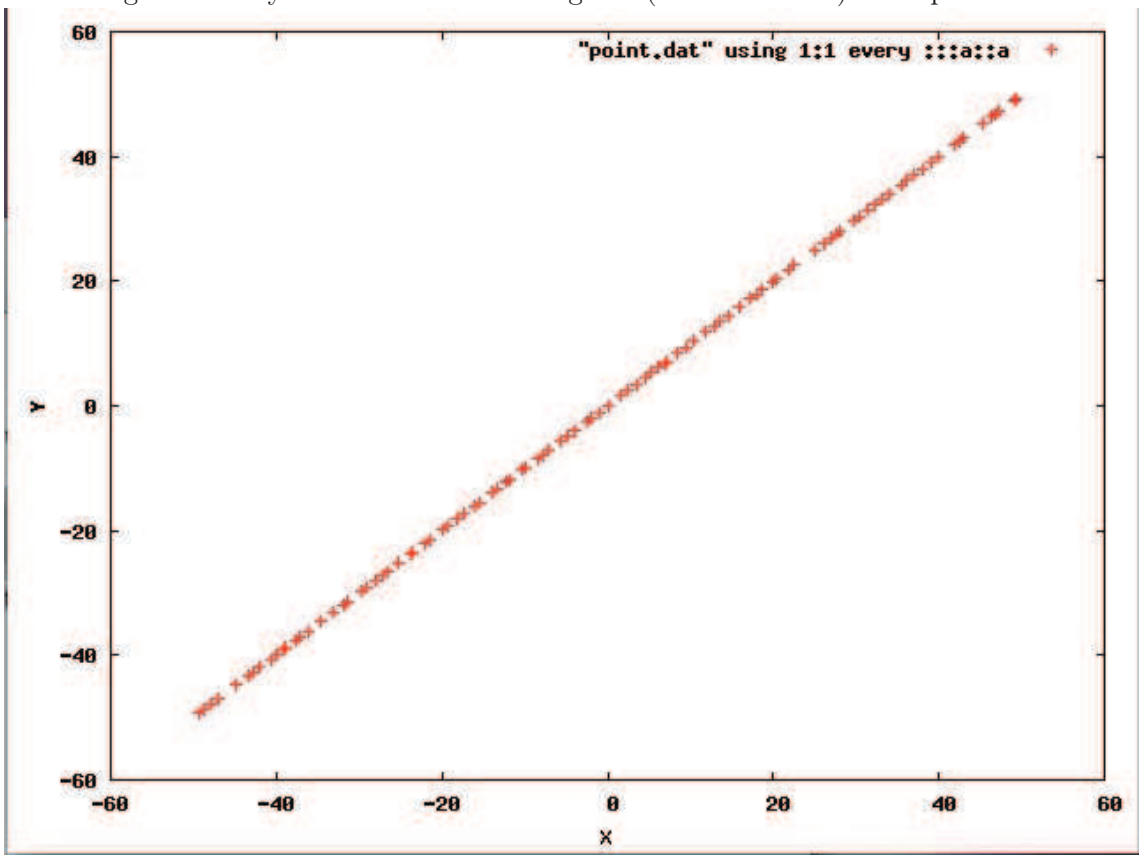


Figure 2.2: Uniform vs determinantal sampling.

Figure 2.3: Dyson's model on the diagonal (one dimension) - 100 particles



As we have said, determinantal process model a certain type of physical interactions very well. They have also arisen in another seemingly orthogonal domain : stochastic matrix theory. It is known that the eigenvalues of certain classes of random matrices (as an example, consider orthogonal matrices sampled from the Haar measure) form a determinantal process.

On the contrary of determinantal processes, permanental processes exhibit attraction. They are a generalization of well known Cox processes. Both of these processes can seemingly be used to model quite diverse phenomena. Indeed, this correlation structure seems to fit many realistic observations in finance, networking, epidemiology,...
In this manuscript, we study specifically α-determinantal point processes, while sometimes restricting our attention to the case $\alpha = -1$, which will be known as the determinantal point process. When we find it to be possible, we generalize the results to any point process possessing a Papangelou conditional intensity, as for example in Section 5.4 or Chapter 6.

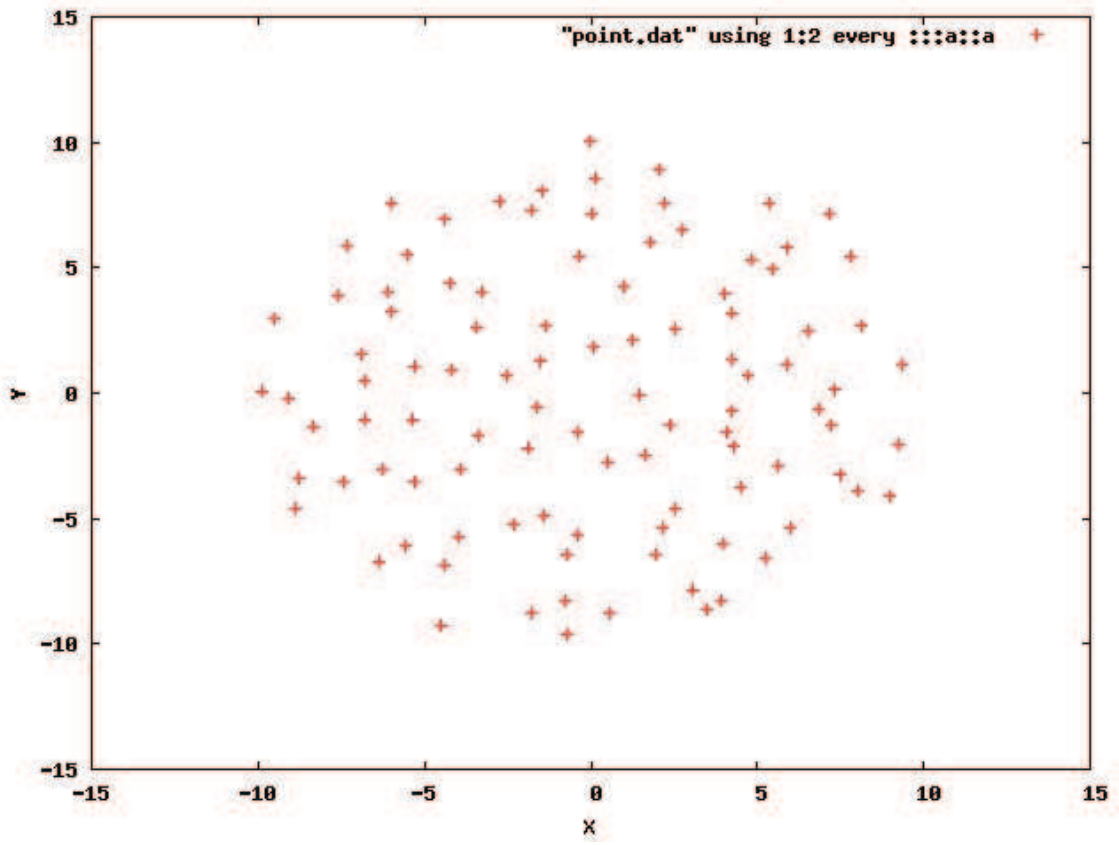
Results

In Chapter 3, we introduce the theoretical background which will be necessary for the study of point processes. After a brief presentation of the classical tools that appear in point process theory, we move on to define integral operators and their kernels. The linear operators introduced there will be central to the study of determinantal point processes, therefore we insist on some important properties of the trace and the Fredholm determinant of such operators. Afterwards, we introduce rigorously α-determinantal point processes by mean of their correlation functions or their Janossy density. Lastly, we insist specifically on the Papangelou conditional intensity which is found to be of most importance in the whole manuscript.

In Chapter 4, we turn our attention to practical ways of simulating α-determinantal point processes, as well as the numerical implementation of models. Indeed, simulation of general determinantal point processes is mostly unexplored, and was in fact initiated in [34] wherein the authors give a practical algorithm for the simulation of determinantal point processes. Theoretical discussion of the aforementioned algorithm as well as statistical aspects have also been explored in [44]. More specifically, the Ginibre point process has spiked interest since its introduction in [30]. Even recently, probabilistic results have been obtained in the specific case of the Ginibre point process, see e.g. [31, 76]. The simulation procedure which is hinted in [30] was fully developed numerically in [13]. To the best of our knowledge, the first use of the Ginibre point process as a model traces back to [12]. More recently, in [54,84,87], different authors have used the Ginibre point process to model phenomena arising in networking. Indeed, this particular model has many advantages with regards to applications. It is indeed invariant with respect to rotations and translations, which gives us a natural compact subset on which to simulate it: the ball centered at the origin. Moreover, the electrostatic repulsion between particles seems to be fitting for many applications. Our aim in this paper is to study the simulation of the Ginibre point process from a practical point of view, and give different methods which will be more or less suited to the application at hand. The main problem that arises in practice is that although the eigenvalues of matrices in the GUE ensemble form a Ginibre point process, these eigenvalues are *not* compactly supported, although after renormalization, they tend to be compactly supported as N tends to infinity (this is known as the circular law in stochastic matrix theory). Moreover, as will be seen here, truncating to a natural compact and letting N tend to infinity is not the most efficient way to proceed, even though this operation preserves the determinantal property of the point process. Therefore, our methods will rely on the modification of the kernel associated with the Ginibre point process. We study in depth the projection of the kernel onto a compact, its truncation to a finite rank, and in the last part a combination of both operations. Each of these operations on the

kernel will have different results on the resulting point process, as well as the simulation techniques involved.
To be more specific concerning our results, a simulation of a rank N (truncated) Ginibre point process is obtained here in Figure 2.4. This point process has N points almost surely, but is not necessarily compactly supported, even though it tends to be as N tends to infinity. Determinantal point process allow a finer control of such properties of the model: we are able to modify the kernel K and introduce (4.15) in order to obtain a compactly supported point process with N points almost surely, and which tends to the Ginibre as N tends to infinity, as proved in Theorem 4.4.1. In effect, we truncated the kernel so that its rank is less than N, then conditioned on there being N points. The conditioning conserves the determinantal property of the point process, and yields a simple model usable in applications such as the one given in [87], and recalled in Appendix A.

Figure 2.4: Rank N Ginibre model in two dimensions - 100 particles



In Chapter 5, we begin by recalling all the notions and properties from Dirichlet form theory which will be useful in the chapter. These notions range from basic properties of Dirichlet form theory well presented in [27] to more advanced results from [49] which will be useful in our analysis. Next, we proceed as in [2] and introduce the same differential gradient alongside their test functionals. More precisely, we say that a function $F : \mathcal{X} \to \mathbb{R}$

is in $\mathcal{F}C_b^\infty(C^\infty(E),\mathcal{X})$ if F is of the form

$$F:\xi\longmapsto f(\sum_{x\in\xi}h_1(x),\dots,\sum_{x\in\xi}h_N(x)),$$

for some $N\in\mathbb{N}$, $h_1,\dots,h_N\in\mathcal{D}=C^\infty(E)$, $F\in C_b^\infty(\mathbb{R}^N)$. For a function F in $\mathcal{F}C_b^\infty(C^\infty(E),\mathcal{X})$, we define its gradient as

$$\nabla F(\xi)=\sum_{i=1}^N\partial_i f(\sum_{x\in\xi}h_1(x),\dots,\sum_{x\in\xi}h_N(x))\sum_{x\in\xi}\nabla^E h_i(x),$$

where ∇^E is the usual gradient on E. Let us now recall some of the results of [2] obtained in the case of the Poisson point process. It was shown there that there exists an integration by parts formula in $\mathrm{L}^2(\mathcal{X},\mu)$ in the sense that there exists an operator, which we call the divergence div such that the following holds:

$$\mathbb{E}[F\,\nabla G]=\mathbb{E}[G\,\mathrm{div}F],$$

for $F,G\in\mathcal{F}C_b^\infty(C^\infty(E),\mathcal{X})$.Then, after the study of the generator $\mathcal{H}F=\mathrm{div}\nabla F$, it is shown that there exists a quasi-continuous diffusion correctly associated with the Poisson point process. In that case, the diffusion is known as a brownian motion on the configuration space. The same study was performed in [3] for the Gibbs point process, and the study was begun in the α-determinantal case in [14] wherein an integration by parts formula was obtained. Our aim in a first step is to clarify some of the arguments in [14] which were found to not be precise. Then, we conduct the rest of the work done in [2,3] in the determinantal case, i.e. we construct the diffusion associated with the determinantal point process. We now explain in detail this part of the chapter, which lead to the publication [20].

We construct Dirichlet forms related to determinantal processes, and we apply them to derive the existence of the associated interacting diffusion processes. For this we provide an integration by parts formula for functionals of determinantal processes which is based on a quasi-invariance result proved in [14]. This integration by parts formula is extended to closed gradient and divergence operators. Our approach follows the lines of [2] and our construction differs from the one considered in [83] which is based on sample-path identities. Such a construction can be applied to derive formulas for density estimation and sensitivity analysis for functionals of determinantal processes along the lines of [69].

Our main result, Theorem 5.3.1, provides the symmetric Dirichlet form associated to a determinantal process. An application to the associated stochastic dynamics follows in Theorem B.5.1, in which we prove the existence of the diffusion process associated with a determinantal process satisfying the assumptions of Theorem 5.3.2.

In the last section of this chapter, we put to light the fundamental specificities of μ that allow there to be an integration by parts formula for the point process. That is, we consider a general point process which we only assume to possess a conditional Papangelou intensity. Then, under some integrability conditions, we show the existence of the associated diffusions. This allows us along the way to slightly generalize the results obtained in [14], in that we include the case of any determinantal point process possessing a conditional Papangelou intensity, i.e. all examples of [28].

Our aim in Chapter 6 is to obtain moment formulae for quite general point processes. Hence, we follow the path of [68], in which some moment formulae were obtained for the Poisson point process. In particular, the main result of [68] is the following formula, obtained in the Poisson case:

$$\mathbb{E}[\big(\sum_{y\in\mathcal{X}} u_y(\xi)\big)^n] = \sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_n} \mathbb{E}\Big[\int_{E^k} u_{x_1}^{|P_1|}\dots u_{x_k}^{|P_k|} \quad (\xi\cup x_1\cup\dots\cup x_k)\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_k)\Big],$$

where $\mathcal{T}_n$ is the set of all partitions of $\{1,\dots,n\}$, $|P_i|$ is the cardinality of P_i, $i=1,\dots,k$, and $u : E\times\mathcal{X}\to\mathbb{R}$ is a nonnegative measurable process.
The proofs in [68] are mostly based on the use of previous results related to Malliavin calculus (in particular the formula that gives $\mathbb{E}[\delta(u)^n]$). In this chapter, we generalize all the formulae in [68] to the case of a point processes which has Papangelou intensities (which obviously includes the case of the Poisson point process). Our proofs are mainly based on the Georgii-Nguyen-Zessin formula, and as a consequence, we also obtain analogues of the formula that gives $\mathbb{E}[\delta(u)^n]$, for a suitable definition of $\delta(u)$ in our context. We also obtain an analogue of the Skohorod isometry formula in a general setting as a nice corollary. In all our results, the difference between the Poissonian case and a general point process is a randomization of the underlying measure λ obtained by multiplying it by $c(\cdot,\xi)$.
Our results also allow us to study random transformations of point processes. In the case of a general point process μ, we consider a random transformation τ, such that each particle x of the configuration ξ is moved to $\tau(x,\xi)$. Then, we obtain an explicit characterization of $\tau^*\mu$ if we assume that τ satisfies a suitable condition, where here τ^* is the image measure of μ. An application is to show that a non-random transformation of a determinantal point process yields another determinantal point process.

Lastly, in the appendices, we include collaborations that do not fit directly in the scope of this dissertation thesis. All three appendices are mostly unedited excerpts from the papers which were the fruit of these collaborations. We therefore just give here a brief summary of the work done therein.
In Appendix A, we present an algorithm for the recovery of wireless networks after a disaster. Considering a damaged wireless network, presenting coverage holes or/and many disconnected components, we propose a disaster recovery algorithm which repairs the network. It provides the list of locations where to put new nodes in order to patch the coverage holes and mend the disconnected components. In order to do this we first consider the simplicial complex representation of the network, then the algorithm adds supplementary vertices in excessive number, and afterwards runs a reduction algorithm in order to reach an optimal result. One of the novelty of this work resides in the proposed method for the addition of vertices. We use a determinantal point process: the Ginibre point process which has inherent repulsion between vertices, and has never been simulated before for wireless networks representation. We compare both the determinantal point process addition method with other vertices addition methods, and the whole disaster recovery algorithm to the greedy algorithm for the set cover problem.
In Appendix B, we construct the interacting diffusion processes associated to determinantal processes. The notations of this paper are quite different than those of the present manuscript, therefore we start by recalling notations and notions specific to this chapter.

Our construction is based on the notion of Dirichlet form and an integration by parts formula for functionals of determinantal processes. We also give examples of such diffusions. In Appendix C, we investigate the application of perfect simulation via coupling from the past on determinantal point processes. We give a general framework for perfect simulation in the determinantal model. It is shown that the limiting sequence of the time-to-coalescence of the coupling is bounded by $K|\Lambda| \log K|\Lambda|$. An application is given to stationary determinantal point processes.

Publications

— Laurent Decreusefond, Ian Flint, Nicolas Privault, Giovanni Luca Torrisi. *Determinantal processes: a survey.*
To be published in "Stochastic analysis for Poisson point processes: Malliavin calculus, Wiener-Itô chaos expansions and stochastic geometry", G. Peccati and M. Reitzner editors, Bocconi & Springer Series, 2014.

— Laurent Decreusefond, Ian Flint. *Moment formulae for general point processes.* Accepted in "Comptes rendus Mathématique", 2013.

Preprints

— Laurent Decreusefond, Ian Flint, Anaïs Vergne. *Efficient simulation of the Ginibre point process.* Submitted, 2013.
http://hal.archives-ouvertes.fr/hal-00869259
Included in Chapter 4.

— Laurent Decreusefond, Ian Flint. *Moment formulae for general point processes.* Submitted to JFA, 2012.
http://hal.archives-ouvertes.fr/hal-00753801
Included in Chapter 6.

— Anaïs Vergne, Ian Flint, Laurent Decreusefond and Philippe Martins. *Homology based algorithm for disaster recovery in wireless networks.* Submitted, 2013.
http://hal.archives-ouvertes.fr/hal-00800520
Included in Appendix A.

— Laurent Decreusefond, Ian Flint, Nicolas Privault, Giovanni Luca Torrisi. *Stochastic dynamics of determinantal processes by integration by parts.* Submitted, 2013.
http://arxiv.org/abs/1210.6109
Included in Appendix B, see also Section 5.3

— Laurent Decreusefond, Ian Flint, Kah Choon Low. *Perfect Simulation of Determinantal Point Processes.* Submitted, 2013.
http://hal.archives-ouvertes.fr/hal-00879101
Included in Appendix C.

Chapter 3

Introduction to the theory of point processes

The general framework was initially developed in [37], see also [17, 18, 64].

3.1 Locally finite point processes

Let E be a Polish space, $\mathcal{O}(E)$ the family of all non-empty open subsets of E and $\mathcal{B}$ denotes the corresponding Borel σ-algebra. For any subset $A \subset E$, let $|A|$ denote the cardinality of A, setting $|A| = \infty$ if A is not finite. We use the notation $\Lambda \subseteq E$, to denote a compact set Λ in E. We denote by $\mathcal{X}$ the set of locally finite point configurations on E:

$$\mathcal{X} := \{\xi \subset E \ : \ |\xi \cap \Lambda| < \infty \quad \forall \Lambda \subseteq E\}.$$

In fact, $\mathcal{X}$ consists of all *simple* positive integer-valued Radon measures on E (by *simple* we mean that for all $x \in E$, $\xi(x) \leq 1$). Sometimes, we write $\mathcal{N}$ for the space of (non-*simple*) positive integer-valued Radon measures on E. Hence, it is naturally topologized by the vague topology, which is the weakest topology such that for all continuous and compactly supported functions f on E, the mapping

$$\xi \mapsto \langle f, \xi \rangle := \sum_{y \in \xi} f(y)$$

is continuous. We denote by $\mathcal{F}$ the corresponding σ-algebra, i.e.

$$\mathcal{F} := \sigma(\{\xi \in \mathcal{X} \ : \ |\xi \cap \Lambda| = m\}, m \in \mathbb{N}, \Lambda \subseteq E \ ).$$

We usually call elements of $\mathcal{X}$ configurations and identify a locally finite configuration $\xi \in \mathcal{X}$ with the atomic Radon measure $\sum_{y \in \xi} \varepsilon_y$, where we have written ε_y for the Dirac measure at $y \in E$. For a given $\xi = \sum_{y \in \xi} \varepsilon_y$, we usually view ξ as a set, and write $\xi \cup y_0 = \xi \cup \{y_0\}$ for the addition of a particle at y_0 and $\xi \setminus y_0 = \xi \setminus \{y_0\}$ for the removal of a particle at y_0. We define similarly $\mathcal{X}_0$ the set of finite point configurations on E:

$$\mathcal{X}_0 := \{\xi \subset E: \ |\xi| < \infty\},$$

which is naturally equipped with the trace σ-algebra $\mathcal{F}_0 = \mathcal{F}|_{\mathcal{X}_0}$. Lastly, for any compact subset $\Lambda \subseteq E$, let $\mathcal{F}_\Lambda$ be the space of finite configurations on Λ, and $\mathcal{F}_\Lambda$ the associated (trace-) σ-algebra.

As in [28], we define for any Radon measure λ on E the (λ-)sample measure L^λ on $(\mathcal{X}_0, \mathcal{F}_0)$ by

$$\int_{\mathcal{X}_0} f(\alpha)\, L^\lambda(\mathrm{d}\alpha) := \sum_{n\geq 0} \frac{1}{n!} \int_{E^n} f(\{x_1, \dots, x_n\})\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n), \tag{3.1}$$

for any measurable $f : \mathrm{N}_f \to \mathbb{R}$. Similarly, we also define its restriction to $\Lambda \subseteq E$ a compact subset:

$$\int_{\mathcal{X}_0} f(\alpha)\, L^\lambda_\Lambda(\mathrm{d}\alpha) := \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} f(\{x_1, \dots, x_n\})\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n),$$

for any measurable $f : \mathcal{X}_\Lambda \to \mathbb{R}$.

A random point process is defined as a probability measure μ on $(\mathcal{X}, \mathcal{F})$. Such a probability measure μ is characterized by its Laplace transform $\mathcal{L}_\mu$, which is defined for any measurable nonnegative function f on E as

$$\mathcal{L}_\mu(f) = \int_{\mathcal{X}} e^{-\langle f, \xi\rangle}\, \mu(\mathrm{d}\xi).$$

With the classical abuse of notation, we scarcely write ξ to denote the canonical random variable on $(\mathcal{X}, \mathcal{F}, \mu)$. In that case, we write $\mathbb{E}$ for the expectation of the random variable ξ, i.e.

$$\mathbb{E}[F(\xi)] = \int_{\mathcal{X}} F(\xi)\, \mu(\mathrm{d}\xi),$$

for any measurable $F : \mathcal{X} \to \mathbb{R}$. We write

$$\xi_\Lambda = \xi \cap \Lambda$$

for the restriction of ξ to a compact subset $\Lambda \subseteq E$. The law of ξ_Λ (i.e. the restriction of μ to $\Lambda \subseteq E$) is denoted by μ_Λ. Lastly, for a compact subset $\Lambda \subseteq E$, we denote by $\xi(\Lambda)$ the number of points of ξ_Λ, i.e. $\xi(\Lambda) := |\xi \cap \Lambda|$.

A point process μ is said to have a correlation function $\rho : \mathcal{X}_0 \to \mathbb{R}$ with respect to (*w.r.t.*) a Radon measure λ on $(E, \mathcal{B})$ if ρ is measurable and

$$\mathbb{E}[\sum_{\alpha\subset\xi,\ \alpha\in\mathcal{X}_0} f(\alpha)] = \int_{\mathcal{X}_0} f(\alpha)\, \rho(\alpha)\, L^\lambda(\mathrm{d}\alpha),$$

for all measurable nonnegative functions f on $\mathcal{X}_0$. When such a measure λ exists, it is known as the intensity measure of μ. For $\alpha = \{x_1, \dots, x_k\}$, where $k \in \mathbb{N}^*$, we sometimes write $\rho(\alpha) = \rho_k(x_1, \dots, x_k)$ and call ρ_k the k-th correlation function, where here ρ_k is a symmetrical function on E^k.

Proposition 3.1.1. *The correlation functions of μ (if they exist),* w.r.t. *a Radon measure λ on E, verify*

$$\mathbb{E}\left[\prod_{i=1}^k \xi(B_i)\right] = \int_{B_1\times\dots\times B_k} \rho_k(x_1, \dots, x_k)\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_k),$$

for any family of mutually disjoint compact subsets $B_1, \dots, B_k$ of E, $k \geq 1$.

Proof. Define u as follows:

$$u = u_N : \omega \mapsto \begin{cases} \mathbf{1}_{\Lambda^N}(\omega) \text{ if } |\omega| = N, \\ 0 \text{ if } |\omega| \neq N. \end{cases}$$

We obtain,

$$\begin{aligned}
\mathbb{E}[\xi(A_1)\dots\xi(A_N)] &= \int_{\mathcal{X}} \sum_{x_1,\dots,x_N \in \xi} \mathbf{1}_{A_1,\dots,A_N}(x_1,\dots,x_N)\,\mu(\mathrm{d}\xi) \\
&= N! \int_{\mathcal{X}} \sum_{\omega \in \mathcal{X}_0 : \omega \subset \xi} u_N(\omega)\,\mu(\mathrm{d}\xi) \\
&= N! \sum_{m=0}^{\infty} \frac{1}{m!} \int_{E^m} u_N(\{x_1,\dots,x_m\}) \rho(\{x_1,\dots,x_m\})\,\lambda^{\otimes m}(\mathrm{d}x_1,\dots,\mathrm{d}x_m) \\
&= \int_{A_1\times\dots\times A_N} \rho(\{x_1,\dots,x_N\})\,\lambda^{\otimes m}(\mathrm{d}x_1,\dots,\mathrm{d}x_N).
\end{aligned}$$

□

The previous formula can be generalized as follows:

Proposition 3.1.2. *Let $A_1,\dots,A_n$ be disjoint bounded Borel subsets of E. Let $k_1,\dots,k_n$ be integers such that $\sum_{i=1}^n k_i = N$. Then,*

$$\mathbb{E}[\prod_{i=1}^{n} \frac{\xi(A_i)!}{(\xi(A_i)-k_i)!}] = \int_{A_1^{k_1}\times\dots\times A_n^{k_n}} \rho(\{x_1,\dots,x_N\})\,\lambda(dx_1)\dots\lambda(dx_N).$$

Proof. The proof is the same as the first proposition. Indeed, Proposition 3.1.2 is the same as Proposition 3.1.1 except for the fact that there are less subsets. This allows us to extend the first formula to subsets that are not disjoint. □

Henceforth, we require in addition that $\rho_k(x_1,\dots,x_k) = 0$ whenever $x_i = x_j$ for some $1 \leq i \neq j \leq k$. Recall that ρ_1 is the particle density with respect to λ, and

$$\rho_n(x_1,\dots,x_n)\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_n)$$

is the probability of finding a particle in the vicinity of each x_i, $i = 1,\dots,n$.

For any compact subset $\Lambda \subseteq E$, the Janossy densities of μ, *w.r.t.* a Radon measure λ on E, are (if they exist) measurable functions $j_\Lambda^n : \Lambda^n \to \mathbb{R}$ satisfying for all measurable functions $f : \mathcal{X}_\Lambda \to \mathbb{R}$,

$$\mathbb{E}\left[f(\xi_\Lambda)\right] = \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} f(\{x_1,\dots,x_n\})\, j_\Lambda^n(x_1,\cdots,x_n)\ \lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_n). \tag{3.2}$$

Remark 2. *Here, we can notice that in definition (3.2), we do not need to make the subscript n directly appear. In fact, we sometimes also use the simplified notation, $j_\Lambda(\alpha) := j_\Lambda^k(x_1,\dots,x_k)$, for $\alpha = \{x_1,\dots,x_k\}$, where $k \in \mathbb{N}^*$.*

Hence, j_Λ is the density of μ_Λ with respect to L^λ_Λ, when $\mu_\Lambda \ll L^\lambda_\Lambda$. For $n \geq 1$, the Janossy densities satisfy the following properties:

— Symmetry:

$$j^n_\Lambda\left(x_{\sigma(1)}, \cdots, x_{\sigma(n)}\right) = j^n_\Lambda\left(x_1, \cdots, x_n\right),$$

for every permutation σ of $\{1, \cdots, n\}$.

— Normalization constraint. For each compact subset $\Lambda \subseteq E$,

$$\sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\Lambda^n} j^n_\Lambda\left(x_1, \cdots, x_n\right)\, \lambda(\mathrm{d}x_1) \ldots \lambda(\mathrm{d}x_n) = 1.$$

For $n \geq 1$, the Janossy density $j^n_\Lambda(x_1, \ldots, x_n)$ is in fact the joint density (multiplied by a constant) of the n points given that the point process has exactly n points. For $n = 0$, $j^0_\Lambda(\emptyset)$ is the probability that there are no points in Λ, and is thus sometimes called the void probability. We also recall that the Janossy densities can be recovered from the correlation functions via the relation

$$j^n_\Lambda(x_1, \ldots, x_n) = \sum_{m \geq 0} \frac{(-1)^m}{m!} \int_{\Lambda^m} \rho_{n+m}(x_1, \ldots, x_n, y_1, \ldots, y_m)\, \lambda(\mathrm{d}y_1) \ldots \lambda(\mathrm{d}y_m),$$

for any compact $\Lambda \subseteq E$, and any $x_1, \ldots, x_n \in \Lambda$. The previous relation can also be inverted, and leads to

$$\rho_n(x_1, \ldots, x_n) = \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} j^{m+n}_\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_m)\, \lambda(\mathrm{d}y_1) \ldots \lambda(\mathrm{d}y_m).$$

The proof of the previous results can be found in [17].

3.2 Kernels and integral operators

From now on, we consider a general Radon measure λ on $(E, \mathcal{B})$. For any compact set $\Lambda \subseteq E$, we denote by $\mathrm{L}^2(\Lambda, \lambda)$ the Hilbert space of complex-valued square integrable functions *w.r.t.* the restriction of the Radon measure λ to Λ, equipped with the inner product

$$< f, g >_{\mathrm{L}^2(\Lambda,\lambda)} := \int_\Lambda f(x)\overline{g(x)}\, \lambda(\mathrm{d}x), \quad f, g \in \mathrm{L}^2(\Lambda, \lambda)$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. By definition, a kernel K is a measurable function from E^2 to $\mathbb{C}$. We say that K is locally square integrable if, for any compact set $\Lambda \subseteq E$, we have

$$\int_{\Lambda^2} |K(x, y)|^2 \lambda(\mathrm{d}x)\lambda(\mathrm{d}y) < \infty.$$

To any locally square integrable kernel K, we associate the integral operators

$$\mathcal{K}_\Lambda : \mathrm{L}^2(\Lambda, \lambda) \to \mathrm{L}^2(\Lambda, \lambda),$$

where Λ is a compact subset of E, defined by

$$\mathcal{K}_\Lambda f(x) := \int_\Lambda K(x, y) f(y)\, \lambda(\mathrm{d}y), \quad \text{for } \lambda\text{-almost all } x \in \Lambda.$$

A straightforward application of the Cauchy-Schwarz inequality shows that the operator $\mathcal{K}_\Lambda$ is bounded when the kernel K is locally square integrable. In fact, it can be shown that $\mathcal{K}_\Lambda$ is a compact operator.

To any locally square integrable kernel K, we also associate the integral operator $\mathcal{K}$ defined by

$$\mathcal{K}f(x) := \int_E K(x,y)f(y)\,\lambda(\mathrm{d}y), \quad \text{for } \lambda\text{-almost all } x \in E$$

for functions $f \in \mathrm{L}^2(E,\lambda)$ that vanish outside a compact subset of E. We note $\mathcal{P}_\Lambda$ the projection operator from $\mathrm{L}^2(E,\lambda)$ to $\mathrm{L}^2(\Lambda,\lambda)$. Therefore, the operator $\mathcal{K}_\Lambda$ defined previously is the projection of $\mathcal{K}$ onto $\mathrm{L}^2(\Lambda,\lambda)$, or equivalently $\mathcal{K}_\Lambda = \mathcal{P}_\Lambda \mathcal{K} \mathcal{P}_\Lambda$. We also denote by K_Λ the kernel of $\mathcal{K}_\Lambda$, i.e. $K_\Lambda(x,y) := \mathbf{1}_\Lambda(x)K(x,y)\mathbf{1}_\Lambda(y)$, for $x, y \in E$. The operator $\mathcal{K}$ is said to be Hermitian or self-adjoint if its kernel verifies

$$K(x,y) = \overline{K(y,x)}, \quad \text{for } \lambda^{\otimes 2}\text{-almost all } (x,y) \in E^2. \tag{3.3}$$

Equivalently, this means that the integral operators $\mathcal{K}_\Lambda$ are self-adjoint for any compact set $\Lambda \subseteq E$. If $\mathcal{K}_\Lambda$ is self-adjoint, by the spectral theorem for self-adjoint and compact operators we have that $\mathrm{L}^2(\Lambda,\lambda)$ has an orthonormal basis $\{\varphi_j^\Lambda\}_{j\geq 1}$ of eigenvectors of $\mathcal{K}_\Lambda$. The corresponding eigenvalues $\{\lambda_j^\Lambda\}_{j\geq 1}$ have finite multiplicity (except possibly the zero eigenvalue) and the only possible accumulation point of the eigenvalues is the zero eigenvalue. Then, the kernel K_Λ of $\mathcal{K}_\Lambda$ can be written as

$$K_\Lambda(x,y) = \sum_{n\geq 1} \lambda_n^\Lambda \varphi_n^\Lambda(x)\overline{\varphi_n^\Lambda(y)}, \tag{3.4}$$

for $x, y \in \Lambda$. We say that an operator $\mathcal{K}$ is positive (respectively nonnegative) if its spectrum is included in $(0,+\infty)$ (respectively $[0,+\infty)$). For two operators $\mathcal{K}$ and $\mathcal{I}$, we say that $\mathcal{K} > \mathcal{I}$ (respectively $\mathcal{K} \geq \mathcal{I}$) in the operator ordering if $\mathcal{K} - \mathcal{I}$ is a positive operator (respectively nonnegative operator).

We say that a self-adjoint integral operator $\mathcal{K}_\Lambda$ is trace-class if

$$\|\mathcal{K}_\Lambda\|_1 := \sum_{n\geq 1} |\lambda_\Lambda^n| < \infty,$$

and we call $\|\mathcal{K}_\Lambda\|_1$ the trace norm of $\mathcal{K}_\Lambda$. We then define the trace of the operator $\mathcal{K}_\Lambda$ as $\mathrm{Tr}\,\mathcal{K}_\Lambda = \sum_{n\geq 1} \lambda_\Lambda^n$. If $\mathcal{K}_\Lambda$ is trace-class for every compact subset $\Lambda \subseteq E$, then we say that $\mathcal{K}$ is locally trace-class. It is easily noted that if a Hermitian integral operator $\mathcal{K} : \mathrm{L}^2(E,\lambda) \to \mathrm{L}^2(E,\lambda)$ is trace-class, then $\mathcal{K}^n$ is also trace class for all $n \geq 2$. In fact, we even have $\mathrm{Tr}(\mathcal{K}^n) \leq \|\mathcal{K}\|^{n-1}\mathrm{Tr}(\mathcal{K})$, where $\|\mathcal{K}\|$ is the operator norm of $\mathcal{K}$.

The practical computations of fractional powers of Fredholm determinants involve the so-called α-determinants, which we introduce now. Take $\alpha \leq 0$. For a square matrix $A = (A_{ij})_{1\leq i,j\leq n}$ of size $n \times n$, the α-determinant $\det_\alpha A$ is defined by:

$$\det_\alpha A = \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^n A_{i\sigma(i)}, \tag{3.5}$$

where S_n stands for the n-th symmetric group and $\nu(\sigma)$ is the number of cycles in the permutation $\sigma \in S_n$. The previous definition actually extends that of the determinant,

which is obtained by taking $\alpha = -1$. Now, we can define the Fredholm determinant of $\mathrm{I}+\mathcal{K}$ as

$$\mathrm{Det}(\mathrm{I}+\mathcal{K}) = \exp\left(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\mathrm{Tr}(\mathcal{K}^n)\right), \tag{3.6}$$

since $\mathrm{Tr}(\mathcal{K}^n) < \infty$. Here, I denotes the identity operator on $\mathrm{L}^2(E,\lambda)$ and $\mathcal{K}$ is a general trace class operator on $\mathrm{L}^2(E,\lambda)$. Then, the fractional powers of Fredholm determinants can be calculated as follows. For any trace class integral operator $\mathcal{K}$, we have

$$\mathrm{Det}(\mathrm{I}-\alpha\mathcal{K})^{-1/\alpha} = \sum_{n\geq 0}\frac{1}{n!}\int_{E^n}\det_\alpha(K(x_i,x_j))_{1\leq i,j\leq n}\,\lambda(\mathrm{d}x_1)\ldots\lambda(\mathrm{d}x_n), \tag{3.7}$$

where K is the kernel of $\mathcal{K}$ and $|\alpha| \leq 1$. (3.7) was obtained in Theorem 2.4 of [77].

Let us end this section by recalling the following result from [28, Lemma A.4]:

Proposition 3.2.1. *Let $\mathcal{K}$ be a nonnegative, bounded, and locally of trace class integral operator on $\mathrm{L}^2(E,\lambda)$. Then one can choose its integral kernel K (defined everywhere) such that the following properties hold:*

(i) K is nonnegative, in the sense that for any $c_1,\ldots,c_n \in \mathbb{C}$, a.e. $x_1,\ldots,x_n \in E$, we have $\sum_{i,j=1}^n \bar{c}_i K(x_i,x_j)c_j \geq 0$.

(ii) K is a Carleman kernel, in the sense that $K_x = K(\cdot,x) \in \mathrm{L}^2(E,\lambda)$ for a.e. $x \in E$.

(iii) For any compact subset $\Lambda \subseteq E$, $\mathrm{Tr}\,\mathcal{K}_\Lambda = \int_\Lambda K(x,x)\,\lambda(\mathrm{d}x)$ and

$$\mathrm{Tr}\,(\mathcal{P}_\Lambda\mathcal{K}^k\mathcal{P}_\Lambda) = \int_\Lambda < K_x, \mathcal{K}^{k-2}K_x >_{\mathrm{L}^2(\Lambda,\lambda)}\ \lambda(\mathrm{d}x),$$

for $k \geq 2$.

Henceforth, under proper hypotheses on $\mathcal{K}$, its kernel is chosen according to the previous proposition.

3.3 α-determinantal point processes

Let

$$\mathbf{M} = \{\alpha \leq 0\ :\ \exists m \in \mathbb{N},\ \alpha = -\frac{1}{m+1}\} \cup \{0\}.$$

With the previous definitions in mind, we move onto the precise definition of α-determinantal point processes. To that effect, we henceforth use the following set of hypotheses:

Hypothesis 3. *Assume that $\alpha \in \mathbf{M}$. Moreover, assume that the map $\mathcal{K}$ is an Hilbert-Schmidt operator from $\mathrm{L}^2(E,\lambda)$ into $\mathrm{L}^2(E,\lambda)$ which satisfies the following conditions:*

i) $\mathcal{K}$ is a bounded symmetric integral operator on $\mathrm{L}^2(E,\lambda)$, with kernel $K(.,.)$.

ii) The spectrum of $\mathcal{K}$ is included in $[0,-\frac{1}{\alpha}]$, , i.e. $0 \leq \mathcal{K} < -\frac{1}{\alpha}\,\mathrm{I}$ in the operator ordering.

iii) The map $\mathcal{K}$ is locally of trace-class.

Suppose that $\mathcal{K}$ satisfies Hypothesis 3. A locally finite and simple point process μ on E is called an α-determinantal process if its correlation functions *w.r.t.* the Radon measure λ on $(E, \mathcal{B})$ exist and satisfy

$$\rho_k(x_1, \dots, x_k) = \det_\alpha(K(x_i, x_j))_{1 \le i,j \le k},$$

for any $k \geq 1$ and $x_1, \dots, x_k \in E$, and where the α-determinant was defined in (3.5). When $\alpha = -1$, we say that μ is a determinantal point process, since the α-determinant reduces to the usual determinant. When $\alpha \neq -1$, μ is called an α-determinantal point process. It is worth noting that under Hypothesis 3, we can choose a proper kernel for $\mathcal{K}$, in the sense of proposition 3.2.1. Clearly, we then have that $\rho_k(x_1, \dots, x_k) \geq 0$ for λ -a.e. $x_1, \dots, x_k \in E$.
In general, we write $\mu^\alpha_{\mathcal{K},\lambda}$ for the α-determinantal point process with integral operator $\mathcal{K}$ and intensity measure λ on E. We omit the superscript α to denote the determinantal point process with integral operator $\mathcal{K}$ and intensity measure λ.

Existence and uniqueness (in law) of determinantal processes (here, $\alpha = -1$) is guaranteed under Hypothesis 3 by the results in [53], [77] and [80]. See also Lemma 4.2.6 and Theorem 4.5.5 in [35]. More precisely, if a kernel K and its associated integral operator $\mathcal{K}$ satisfy Hypothesis 3, then there exists a determinantal process $\mu_{\mathcal{K},\lambda}$ on E associated to $\mathcal{K}$. Moreover, for any compact subset $\Lambda \subseteq E$ there exist constants $c_1(\Lambda), c_2(\Lambda) > 0$ such that $\mu_{\mathcal{K},\lambda}(\xi(\Lambda) > k) \leq c_1(\Lambda) \mathrm{e}^{-c_2(\Lambda)k}$ for all $k \geq 1$, and in this case the correlation functions $\rho_k(x_1, \dots, x_k)$ uniquely determine the law of the process. This is because of the fact that for disjoint compact subsets $B_1, \dots, B_k \subseteq E$, the random vector $(\xi(B_1), \dots, \xi(B_k))$ has a convergent Laplace transform in a neighborhood of zero if the tails of the distributions of $\xi(\Lambda)$ are exponential, cf. [35, Remark 1.2.4]. Another perhaps quicker way to prove the uniqueness of the resulting point process is to use the general criterion originally derived in [45] which ensures uniqueness:

$$\sum_{k \geq 0} \Big(\frac{1}{k!} \int_{\Lambda^k} \rho_k(x_1, \dots, x_k) \, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_k) \Big)^{-1/k} = +\infty,$$

for any Borel set Λ. To prove that the previous series diverges, one can use the classical Fredholm formula (see [79]) which states

$$\frac{1}{k!} \int_{\Lambda^k} \det(K(x_i, x_j))_{1 \le i,j \le k} \, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_k) = \mathrm{Tr}\,(\wedge^k \mathcal{K}_\Lambda),$$

where we have defined $\wedge^k \mathcal{T} := \mathcal{T} \otimes \cdots \otimes \mathcal{T}|_{\mathcal{A}_s(\mathrm{L}^2(E,\lambda)^{\otimes k})}$, for any operator $\mathcal{T}$ on $\mathrm{L}^2(E, \lambda)$, and where $\mathcal{A}_s(H)$, for H a separable Hilbert space, is its anti-symmetric subspace. Then, it suffices to use the estimate $\mathrm{Tr}\,(\wedge^k \mathcal{K}_\Lambda) \leq \frac{1}{k!} \mathrm{Tr}\,(\wedge \mathcal{K}_\Lambda)^k$ which is proved eg. in [79]. This yields

$$\Big(\frac{1}{k!} \int_{\Lambda^k} \rho_k(x_1, \dots, x_k) \, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_j) \Big)^{-1/k} \geq (k!)^{1/k} \mathrm{Tr}\,(\wedge \mathcal{K}_\Lambda)^{-1} \xrightarrow[k \to \infty]{} +\infty,$$

which proves uniqueness.

For $\alpha \in \mathbf{M}$, existence and uniqueness (in law) of the α-determinantal processes is guaranteed under Hypothesis 3 by the results in [77] for α-determinantal point processes, see

also [53] and [80] for the determinantal case. Essentially, existence and uniqueness is due to a consistency condition verified by the Janossy densities. However, it was proved in [8] that for $\alpha \notin \mathbf{M}$, the associated determinantal process does not exist. More precisely,

Theorem 3.3.1. *For $\alpha \notin \mathbf{M}$, we do not have that for all real symmetric positive semidefinite A,*

$$\det_\alpha A \geq 0,$$

and as a consequence, the correlation functions are not correctly defined for $\alpha \notin \mathbf{M}$.

From now on, we therefore consider $\alpha \in \mathbf{M}$, and assume that the integral operator $\mathcal{K}$ satisfies Hypothesis 3. Let us now recall the following result from [77] that gives the Laplace transform of $\mu^\alpha_{\mathcal{K},\lambda}$.

Theorem 3.3.2. *Let $\mathcal{K}$ be an operator satisfying assumption Hypothesis 3. Then, the unique α-determinantal point process $\mu^\alpha_{\mathcal{K},\lambda}$ has Laplace transform:*

$$\mathcal{L}_{\mu^\alpha_{\mathcal{K},\lambda}}(f) = \mathrm{Det}\left(\mathrm{I}+\alpha\mathcal{K}_\varphi\right)^{-1/\alpha},$$

for each nonnegative f on E with compact support, where $\varphi = 1 - e^{-f}$ and $\mathcal{K}_\varphi$ is the trace-class integral operator with kernel

$$K_\varphi(x,y) = \sqrt{\varphi(x)}K(x,y)\sqrt{\varphi(y)}, \quad x,y \in E.$$

Moreover, $\mu^\alpha_{\mathcal{K},\lambda}$ is necessarily simple.

As a consequence of the calculation of the Laplace transform, it is known that $\mu^\alpha_{\mathcal{K},\lambda}$ is a superposition of $m+1$ independent copies of a determinantal point process of kernel $\frac{\mathcal{K}}{m+1}$.

Let $\mathcal{K}$ be an operator satisfying assumption Hypothesis 3. We define the trace class integral operator on $\mathrm{L}^2(E,\lambda)$

$$\mathcal{J}[\Lambda] := (\mathrm{I}+\alpha\mathcal{K}_\Lambda)^{-1}\mathcal{K}_\Lambda,$$

where the compact subset $\Lambda \subseteq E$ indexes the operators $\mathcal{J}[\Lambda]$. $\mathcal{J}[\Lambda]$ is defined in order for $\mathcal{K}$ and $\mathcal{J}[\Lambda]$ to be quasi-inverses in the sense that

$$(\mathrm{I}+\alpha\mathcal{K}_\Lambda)\,(\mathrm{I}-\alpha\mathcal{J}[\Lambda]) = \mathrm{I}\,. \tag{3.8}$$

The operator $\mathcal{J}[\Lambda]$ is called the local interaction operator and we emphasize the fact that unlike $\mathcal{K}_\Lambda$, $\mathcal{J}[\Lambda]$ is not a projection operator, i.e. in general, $\mathcal{J}[\Lambda] \neq \mathcal{P}_\Lambda(\mathrm{I}+\alpha\mathcal{K})^{-1}\mathcal{K}\mathcal{P}_\Lambda$. However, $\mathcal{J}[\Lambda]$ does have some notable properties which are summarized in [28]. Let us give a few of the ones that are the most useful to our purposes. First, it is easily seen that $\mathcal{J}[\Lambda]$ exists as a bounded operator since $\|\alpha\mathcal{K}\| < 1$, and its spectrum is included in $[0,+\infty)$. $\mathcal{J}[\Lambda]$ is again an integral operator, so let us denote by $J[\Lambda]$ its kernel (in fact, one can even show that $\mathcal{J}[\Lambda]$ is also a Carleman operator, cf. [28]). Moreover, since $\mathcal{J}[\Lambda] \leq (1+\|\alpha\mathcal{K}\|)^{-1}\mathcal{K}_\Lambda$, we have that $\mathcal{J}[\Lambda]$ is again a trace operator. For $\alpha = \{x_1,\ldots,x_k\} \in \mathcal{X}_\Lambda$, we denote by $\det_\alpha J[\Lambda](\alpha)$ $(= \det_\alpha J[\Lambda](\{x_1,\ldots,x_k\}))$ the determinant $\det_\alpha\left(J[\Lambda](x_i,x_j)\right)_{1\leq i,j\leq k}$. Note that for all $k \in \mathbb{N}^*$, the function

$$(x_1,\ldots,x_k) \mapsto \det_\alpha J[\Lambda](\{x_1,\ldots,x_k\})$$

is $\lambda^{\otimes k}$-a.e. nonnegative (thanks to Proposition 3.2.1) and symmetric in $x_1,\ldots,x_k$ (see e.g. [28]), and we simply write $\det_\alpha J[\Lambda](\{x_1,\ldots,x_k\}) = \det_\alpha J[\Lambda](x_1,\ldots,x_k)$. The local interaction operator is particularly useful to the study of determinantal processes because of the following proposition:

Proposition 3.3.1 ([77]). *Assume that the kernel $\mathcal{K}$ satisfies Hypothesis 3. Then, the determinantal process $\mu^{\alpha}_{\mathcal{K},\lambda}$ admits Janossy densities j^n_Λ, given for a compact subset $\Lambda \subseteq E$ and $n \in \mathbb{N}^*$ by*

$$j^n_\Lambda(x_1, \dots, x_n) = \mathrm{Det}(\mathrm{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \det_\alpha J[\Lambda](x_1, \dots, x_n), \tag{3.9}$$

for $x_1, \dots, x_n \in \Lambda$. We can also determine the void probability $j^0_\Lambda(\emptyset) = \mathrm{Det}(\mathrm{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha}$. Here, Det *stands for the Fredholm determinant as defined in (3.6).*

Here, it should be noted that (3.9) still makes sense if $\|\mathcal{K}_\Lambda\| = 1$, since the zeros of $\mathrm{Det}(\mathrm{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha}$ are of the same order of the poles of $\det_\alpha J[\Lambda](x_1, \dots, x_k)$, both of these functions being analyzed as functions of $\lambda^1_\Lambda, \dots, \lambda^n_\Lambda, \dots$ the eigenvalues of $\mathcal{K}_\Lambda$. This previous result is proved in detail in [77]. It is also possible to calculate the moment generation function of α-determinantal processes.

Proposition 3.3.2 (Moment generating function of α-permanental processes)**.** *The moment generating function of the number of points in the compact set $\Lambda \subseteq E$ is*

$$\mathrm{M}(u) = \mathbb{E}[e^{u\xi(\Lambda)}] = \mathrm{Det}(\mathrm{I} + \alpha(1 - e^u)K_\Lambda)^{-1/\alpha},$$

for $u \in \mathbb{R}$.

Proof. Recall that for a measurable $F : \mathbb{N} \to \mathbb{R}_+$,

$$\int_{\mathcal{X}_\Lambda} F(\xi(\Lambda))\, \mu(\mathrm{d}\xi) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} F(m) j_\Lambda(\{x_1, \dots, x_m\}) \lambda^{\otimes m}(\mathrm{d}x_1, \dots, \mathrm{d}x_m).$$

In this case, by formula (3.9), we have

$$\mathrm{M}(u) = \mathrm{Det}(\mathrm{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \det_\alpha(e^u J[\Lambda](x_1, \dots, x_m)) \lambda^{\otimes m}(\mathrm{d}x_1, \dots, \mathrm{d}x_m).$$

Here, we recognize the expansion of $\mathrm{Det}(\mathrm{I} - \alpha e^u \mathcal{J}[\Lambda])^{-1/\alpha}$ shown in formula (3.7), and get

$$\mathrm{M}(u) = \mathrm{Det}\left((\mathrm{I} + \alpha\mathcal{K}_\Lambda)(\mathrm{I} - \alpha e^u \mathcal{J}[\Lambda])\right)^{-1/\alpha}.$$

We obtain the result by expanding the product, since by (3.8), we have

$$(\mathrm{I} + \alpha\mathcal{K}_\Lambda)(\mathrm{I} - \alpha e^u \mathcal{J}[\Lambda]) = \mathrm{I} + (\mathrm{I} + \alpha\mathcal{K}_\Lambda)\left(\alpha\mathcal{J}[\Lambda](1 - e^u)\right) = \mathrm{I} + \alpha(1 - e^u)\mathcal{K}_\Lambda.$$

□

Note that $J[\Lambda]$ can be decomposed in the same basis of $\mathrm{L}^2(\Lambda, \lambda)$ as K_Λ. More precisely, as a consequence of (3.4), we obtain

$$J[\Lambda](x, y) = \sum_{n \geq 1} \frac{\lambda^n_\Lambda}{1 + \alpha\lambda^n_\Lambda} \varphi^n_\Lambda(x)\overline{\varphi^n_\Lambda(y)}, \tag{3.10}$$

for $x, y \in \Lambda$.

Let us conclude by giving some properties linking the rank of $\mathcal{K}$ and the number of points in the associated point process.

Proposition 3.3.3 (Theorem 4 in [80], see also [35, 77]). *Let $\mathcal{K}$ be an operator satisfying Hypothesis 3, and consider the case $\alpha = -1$, i.e. $\mu_{\mathcal{K},\lambda}$ is a determinantal process. We have the following properties.*

a) The probability of the event that the number of all particles is finite is either 0 or 1, depending on whether $\mathrm{Tr}(\mathcal{K})$ is finite or infinite. As expected, the number of points in a compact subset $\Lambda \subseteq E$ is finite since $\mathrm{Tr}(\mathcal{K}_\Lambda) < \infty$.

b) The number of particles is less than or equal to $n \in \mathbb{N}^*$ with probability 1 if and only if $\mathcal{K}$ is a finite rank operator satisfying $\mathrm{Rank}(\mathcal{K}) \leq n$.

c) The number of particles is $n \in \mathbb{N}^*$ with probability 1 if and only if $\mathcal{K}$ is an orthogonal projector satisfying $\mathrm{Rank}(\mathcal{K}) = n$.

We also need a simple condition on the kernels to ensure proper convergence of the associated determinantal measure. This is provided by Proposition 3.10 in [77]:

Proposition 3.3.4. *Let $(\mathcal{K}^{(n)})_{n\geq 1}$ be integral operators with nonnegative continuous kernels $K^{(n)}(x,y)$, $x,y \in E$. Assume that $\mathcal{K}^{(n)}$ satisfy Hypothesis 3, $n \geq 1$, and that $K^{(n)}$ converges to a kernel K uniformly on each compact as n tends to infinity. Then, the kernel K defines an integral operator $\mathcal{K}$ satisfying Hypothesis 3. Moreover, for $\alpha \in \mathbf{M}$, the α-determinantal measure $\mu^\alpha_{\mathcal{K}^{(n)},\lambda}$ converges weakly to the measure $\mu^\alpha_{\mathcal{K},\lambda}$ as n tends to infinity.*

Let us conclude this section by mentioning the particular case of the determinantal projection process. We define a projection kernel (onto $\{\varphi_n,\ 0 \leq n \leq N\} \subset \mathrm{L}^2(E,\lambda)$) to be

$$T_p(x,y) = \sum_{n=0}^{N} \varphi_n(x)\overline{\varphi_n(y)}, \quad \forall x,y \in \mathbb{C} \tag{3.11}$$

where $N \in \mathbb{N}$, and $(\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal family of $\mathrm{L}^2(E,\lambda)$. We call the associated α-determinantal process an α-determinantal projection process (onto $\{\varphi_n,\ 0 \leq n \leq N\} \subset \mathrm{L}^2(E,\lambda)$).

3.4 Papangelou conditional intensity

3.4.1 Basic properties

Let us start by defining the so-called Campbell measures, and giving some of its basic properties, as recalled in [28]. The interested reader may also find further details in [18,52].

Definition 4 (Campbell measures). *The reduced Campbell measure of a point process μ is the measure C_μ on the product space $(E \times \mathcal{X}, \mathcal{B} \otimes \mathcal{F})$ defined by*

$$C_\mu(A \times B) = \int \sum_{x\in\xi} \mathbf{1}_A(x)\mathbf{1}_B(\xi \setminus x)\, \mu(d\xi), \tag{3.12}$$

where $A \in \mathcal{B}$ and $B \in \mathcal{F}$. We define similarly the reduced compound Campbell measure of a point process μ as the measure $\hat{C}_\mu$ on the product space $(\mathcal{X}_0 \times \mathcal{X}, \mathcal{F}_0 \otimes \mathcal{F})$ defined by

$$\hat{C}_\mu(A \times B) = \int \sum_{\alpha\subset\xi,\ \alpha\in\mathcal{X}_0} \mathbf{1}_A(\alpha)\mathbf{1}_B(\xi \setminus \alpha)\, \mu(d\xi),$$

where $A \in \mathcal{F}_0$ and $B \in \mathcal{F}$.

We now define the so-called condition (Σ_λ) initially introduced in [63] and [43] as follows:

Hypothesis 4. *We say that the point process μ satisfies condition (Σ_λ) if $C_\mu \ll \lambda \otimes \mu$. Any Radon-Nikodym density c of C_μ relative to $\lambda \otimes \mu$ is called a version of the Papangelou intensity of μ.*

The preceding assumption also implies that $\hat{C}_\mu \ll L^\lambda \otimes \mu$ and we thus similarly denote any Radon-Nikodym density of $\hat{C}_\mu$ relative to $L^\lambda \otimes \mu$ by $\hat{c}$, and call $\hat{c}$ the compound Papangelou intensity of μ. One then has for any $\xi \in \mathcal{X}$, $\hat{c}(\emptyset, \xi) = 1$, as well as for all $x \in E$, $\hat{c}(x, \xi) = c(x, \xi)$. The Papangelou intensity c of μ is interpreted as the conditional density given the configuration ξ. More precisely,

$$c(x, \xi)\, \lambda(\mathrm{d}x)$$

is the probability of finding a particle in the vicinity of $x \in E$ conditionally on the configuration ξ.
The compound Papangelou intensity verifies the following commutation relation:

$$\forall \eta, \nu \in \mathcal{X}_0, \quad \forall \xi \in \mathcal{X}, \quad \hat{c}(\nu, \eta \cup \xi)\, \hat{c}(\eta, \xi) = \hat{c}(\nu \cup \eta, \xi). \tag{3.13}$$

The recursive application of the previous relation also yields $\forall x_1, \ldots, x_n \in E, \forall \xi \in \mathcal{X}$,

$$\hat{c}(\{x_1, \ldots, x_n\}, \xi) = \prod_{k=1}^{n} c(x_k, \xi \cup x_1 \cup \cdots \cup x_{k-1}),$$

where we have used the convention $x_0 := \emptyset$.

Hypothesis 4, along with the definition of the reduced Campbell measure, allows us to write the following important identity, known as the Georgii-Nguyen-Zessin identity:

$$\int_{\mathcal{X}} \sum_{y \in \xi} u(y, \xi \setminus y)\, \mu(\mathrm{d}\xi) = \int_{\mathcal{X}} \int_E u(z, \xi)\, c(z, \xi)\, \lambda(\mathrm{d}z) \mu(\mathrm{d}\xi), \tag{3.14}$$

for all C_μ-measurable nonnegative functions $u : E \times \mathcal{X} \to \mathbb{R}$. We also have a similar identity for the compound Papangelou intensity:

$$\int_{\mathcal{X}} \sum_{\alpha \subset \xi,\ \alpha \in \mathcal{X}_0} u(\alpha, \xi \setminus \alpha)\, \mu(\mathrm{d}\xi) = \int_{\mathcal{X}} \int_{\mathcal{X}_0} u(\alpha, \xi)\, \hat{c}(\alpha, \xi)\, L^\lambda(\mathrm{d}\alpha) \mu(\mathrm{d}\xi), \tag{3.15}$$

for all $\hat{C}_\mu$-measurable nonnegative functions $u : \mathcal{X}_0 \times \mathcal{X} \to \mathbb{R}$. Combining relation (3.14) and the definition of the correlation functions, we find

$$\mathbb{E}[c(x, \xi)] = \rho_1(x), \tag{3.16}$$

for almost every $x \in E$. We also find more generally, using (3.15), that

$$\mathbb{E}[\hat{c}(\alpha, \xi)] = \rho(\alpha), \tag{3.17}$$

for almost every $\alpha \in \mathcal{X}_0$.

3.4.2 Papangelou intensity of determinantal point processes

In this section, we consider the case of $\alpha = -1$ and recall some of the results of [28]. The first main result gives the Papangelou intensity of μ_Λ, the restriction of μ to a compact subset $\Lambda \subseteq E$.

Theorem 3.4.1 (Theorem 3.1 of [28]). *For each $\Lambda \subseteq E$, μ_Λ satisfies condition $(\Sigma_{\lambda_\Lambda})$ (here, λ_Λ is the restriction of λ to the compact set Λ). A version of its compound Papangelou intensity $\hat{c}_\Lambda$ is given by*

$$\hat{c}_\Lambda(\alpha, \xi) = \frac{\det J[\Lambda](\alpha \cup \xi)}{\det J[\Lambda](\xi)}, \quad \alpha \in \mathcal{X}_0,\ \xi \in \mathcal{X}, \tag{3.18}$$

where the ratio is defined to be zero when the denominator vanishes. This version also satisfies the inequalities

$$\hat{c}_\Lambda(\alpha, \xi) \geq \hat{c}_\Lambda(\alpha, \eta), \quad \text{and} \quad 0 \leq \hat{c}_\Lambda(\alpha, \xi) \leq \det J[\Lambda](\alpha) \leq \prod_{x \in \alpha} J[\Lambda](x, x), \tag{3.19}$$

whenever $\xi \subset \eta \in \mathcal{X}_\Lambda$ and $\alpha \subset \mathcal{X}_\Lambda \setminus \eta$.

Let us now define the operator

$$\mathcal{J} := (\mathrm{I} + \alpha \mathcal{K})^{-1} \mathcal{K}, \tag{3.20}$$

which can be thought of as a global interaction operator. As proved in [28], $\mathcal{J}$ satisfies the expected properties: it is a locally trace class, integral operator, and its kernel $(x, y) \mapsto J(x, y)$ can be chosen to satisfy Proposition 3.2.1. Then, we have that the determinantal process is stochastically dominated by the Poisson point process with intensity $x \mapsto J(x, x)\, \lambda(\mathrm{d}x)$, denoted by π_J. This is denoted by

$$\mu \preceq \pi_J, \tag{3.21}$$

and was proved in [28]. See also [29] for further results on stochastic domination. Recall that for two point processes ν, ν', we say that $\nu \preceq \nu'$ if

$$\int f \, \mathrm{d}\nu \leq \int f \, \mathrm{d}\nu',$$

for all increasing measurable f. Here, we say that f is increasing if $f(\eta) \leq f(\eta')$ whenever $\eta \subset \eta' \in \mathcal{X}$.

For our purposes, it remains to recall the main theorem of [28]:

Theorem 3.4.2 (Theorem 3.6 in [28]). *The determinantal point process $\mu_{\mathcal{K},\lambda}$ satisfies condition (Σ_λ), and its compound Papangelou intensity is given by*

$$\hat{c}(\alpha, \xi) = \lim_{n \to \infty} \hat{c}_{\Delta_n}(\alpha, \xi_{\Delta_n}), \quad \text{for } L^\Lambda \otimes \mu \text{ - almost every } (\alpha, \xi)\ , \tag{3.22}$$

where $(\Delta_n)_{n \in \mathbb{N}}$ is any sequence of compacts included in E, that increase to the whole E.

It should be noted that in general, (3.22) does not give a closed form for the compound Papangelou intensities. In order to write $\hat{c}$ in closed form, additional hypotheses need to be assumed, as seen in Proposition 3.9 of [28]. More precisely, we define

Hypothesis 5. *Suppose $E = \mathbb{R}^d$, λ is Lebesgue measure and,*
- *$\mathcal{J}$ has a continuous integral kernel J.*
- *J has finite range $R < \infty$, $J(x, y) = 0$ if $|x - y| \geq R$.*
- *$\mu_{\mathcal{K},\lambda}$ does not percolate.*

Under Hypothesis 5, the following holds

Theorem 3.4.3 (Proposition 3.9 in [28])**.** *Assume that Hypothesis 5 holds. Then, the determinantal point process $\mu_{\mathcal{K},\lambda}$ satisfies condition (Σ_λ), and its compound Papangelou intensity is given by*

$$\hat{c}(\alpha, \xi) = \frac{\det J(\xi_W \cup \alpha)}{\det J(\xi_W)} \mathbf{1}_{\operatorname{diam} W(\alpha,\xi) < \infty}, \qquad \text{for } L \otimes \mu_{\mathcal{K},\lambda} \text{ - almost every } (\alpha, \xi) \ , \tag{3.23}$$

where $W(\alpha, \xi)$ is the union of clusters of $B_R(\alpha \cup \xi)$ hitting α, and $\xi_W := \xi_{W(\alpha,\xi)}$ is the restriction of ξ to $W(\alpha, \xi)$.

Chapter 4

Simulation of α-determinantal point processes

We proceed as follows. We start by recalling the algorithm from [34] as well as some more advanced results from [44] in Section 4.1. Then, in Section 4.2, we present a new method utilizing perfect simulation of point processes developed in [37,41]. In Section 4.3, we present more specifically the Ginibre point process, and prove some probabilistic properties. We discuss the truncation, and the projection of the Ginibre kernel and gives the basic ideas that will yield different simulation techniques.

4.1 Simulation of α-determinantal point processes by projection kernels

The main results of this section can be found in the seminal work of [34], along with the precisions found in [35] and [44]. We recall the algorithm introduced there in order to insist on its advantages and disadvantages compared to directly simulating according to the densities. The idea of the algorithm presented in the previous papers is two-fold. First, it yields a way to simulate the number of points $n \in \mathbb{N}$ of any determinantal process in a given compact set $\Lambda \subset E$. Second, it explicits an efficient algorithm for the simulation of the (unordered) density of the point process, conditionally on there being n points, i.e. it yields an efficient algorithm to simulate according to the density j_Λ^n. Let us now discuss in detail these two steps.

The central theorem of this section is proved in [34, Theorem 7]. We include the trivial generalization to the case of an α-determinantal, when $\alpha = -\frac{1}{m} \in \mathbf{M}$.

Theorem 4.1.1. *Let $\mathcal{K}$ be a trace-class integral operator satisfying Hypothesis 3 (we often take $\mathcal{K}_\Lambda$, which is indeed trace-class) with kernel K given by*

$$K(x,y) = \sum_{n\geq 1} \lambda_n \varphi_n(x)\overline{\varphi_n(y)}, \quad x,y \in E. \tag{4.1}$$

Then, define $(B_n^i)_{n\in\mathbb{N}}$ a series (possibly infinite), indexed by $i \in \{1,\dots,m+1\}$ of independent Bernoulli random variables of mean $\mathbb{E}[B_n^i] = \frac{\lambda_n}{m}$, $n \in \mathbb{N}^*$. The Bernoulli random variables are defined on a distinct probability space, say $(\Omega, \tilde{\mathcal{F}})$, and for $i \neq j$, the family $(B_n^i)_{n\in\mathbb{N}}$ is independent of $(B_n^j)_{n\in\mathbb{N}}$. Then, define $m+1$ random kernels defined for

$1 \leq i \leq m+1$ *by*

$$K_B^i(x,y) = \sum_{n\geq 1} B_n^i \varphi_n(x)\overline{\varphi_n(y)}, \quad x,y \in E.$$

We define the point process η^i on $(\mathcal{X}\times\Omega, \mathcal{F}\otimes\tilde{\mathcal{F}})$ as the point process obtained by first drawing the Bernoulli random variables, and then the point process with kernel K_B^i, independently of the other η^j, $j\neq i$. We then define η on $(\mathcal{X}\times\Omega, \mathcal{F}\otimes\tilde{\mathcal{F}})$ as $\eta := \eta^1\cup\cdots\cup\eta^{m+1}$.

Then, we have that in distribution, η is a determinantal process with kernel K.

For the remainder of this section, we consider a general kernel K of the form (4.1). We wish to obtain a sample of the aforementioned point process.

4.1.1 Number of points

According to Theorem 4.1.1, the law of the number of points on E has the same law as a sum of Bernoulli random variables. More precisely,

$$|\xi(E)| \sim \sum_{n\geq 1}\sum_{i=1}^{m+1} B_n^i,$$

where $B_n^i \sim \mathrm{Be}(\frac{\lambda_n}{m})$, $n\in\mathbb{N}$. Define $T^i := \sup\{n\in\mathbb{N}_* \ / \ B_n^i = 1\}$, for $1\leq i\leq m+1$. Since $\sum_{n\geq 1}\frac{\lambda_n}{m} = \sum_{n\geq 1}\mathbb{P}(B_n^i=1) < \infty$, by a direct application of the Borel-Cantelli lemma, we have that $T^i < \infty$ almost surely. Hence the method is to simulate a realization t of T^i, then conditionally on $T^i = t$, simulate $B_1^i,\dots,B_{t-1}^i$ which are independent of T^i (note here that $B_t^i = 1$ almost surely).

The simulation of the random variable T^i can be obtained by the inversion method, as we know its cumulative distribution function explicitly. Indeed, for $n\in\mathbb{N}$,

$$\mathbb{P}(T^i = n) = \frac{\lambda_n}{m}\prod_{i=n+1}^{\infty}(1-\frac{\lambda_i}{m}),$$

hence

$$F(t) = \mathbb{P}(T\leq t) = \sum_{n\leq t}\frac{\lambda_n}{m}\prod_{i=n+1}^{\infty}(1-\frac{\lambda_i}{m}), \quad \forall t\in\mathbb{N}. \tag{4.2}$$

While it is possible to simulate an approximation of the previous distribution function, this requires a numerical approximation of the infinite product, as well as the pseudo-inverse $F^{-1}(u) = \inf\{t\in\mathbb{N} \ / \ F(t)\geq u\}$. We also note that in many practical cases, as is the case with the Ginibre point process, the numerical calculations of the previous functions may well be tedious.

Now, assume that we have simulated $B_1^i,\dots,B_{t-1}^i,B_t^i$ for $1\leq i\leq m+1$. If we write $I^i := \{1\leq j\leq m \ : \ B_j^i = 1\}$, then Theorem 4.1.1 assures us that it remains to simulate a determinantal point process with kernel $\sum_{i\in I^1\cup\cdots\cup I^{m+1}}\varphi_i(x)\overline{\varphi_i(y)}$, $x,y\in\Lambda$, which has $|I^1|+\cdots+|I^{m+1}|$ points almost surely. This is the aim of the next subsection.

4.1.2 Simulation of the positions of the points

Assume we have simulated the number of points for each of the independent processes $|I^i| = n_i \in \mathbb{N}$ according to the previous subsection. For the clarity of the presentation, we also assume that $B^i_1 = 1, \dots, B^i_{n_i} = 1$, where $(B^i_n)_{n\in\mathbb{N}}$ are the Bernoulli random variables defined previously. This assumption is equivalent to a simple reordering of the eigenvectors $(\varphi^i_n)_{n\in\mathbb{N}}$ (the reordering depends on i). Then, conditionally on there being n_i points, we have reduced the problem to that of simulating the vector $(X^i_1, \dots, X^i_{n_i})$ of joint density

$$p(x_1, \dots, x_n) = \frac{1}{n!} \det\left(\tilde{K}^i(x_i, x_j)\right)_{1\le i,j\le n_i},$$

where $\tilde{K}^i(x, y) = \sum_{j=1}^{n_i} \psi^i_j(x)\overline{\psi^i_j(y)}$, for $x, y \in \Lambda$, where here $(\psi^i_j)_{j\in\mathbb{N}}$ is the reordering of $(\varphi^i_j)_{j\in\mathbb{N}}$. The determinantal point process of kernel $\tilde{K}^i$ has n_i points almost surely, which means that it remains to simulate the unordered vector $(X^i_1, \dots, X^i_{n_i})$ of points of the point process. The idea of the algorithm is to start by simulating $X^i_{n_i}$, then $X^i_{n_i-1}|X^i_{n_i}$, up until $X^i_1|X^i_2, \dots, X^i_{n_i}$. The key here is that in the determinantal case, the density of these conditional probabilities takes a computable form. Let us start by observing, as is used abundantly in [34], that

$$\det\left(\tilde{K}^i(x_i, x_j)\right)_{1\le i,j\le n_i} = \det\left(\psi^i_k(x_l)\right)_{1\le k,l\le n_i} \det\left(\overline{\psi^i_l(x_k)}\right)_{1\le k,l\le n_i},$$

which allows us to visualize the way the algorithm functions. Indeed, the density of X^i_1 is, for $x_1 \in \Lambda$:

$$\begin{aligned}
p_1(x_1) &= \int \dots \int p(x_1, \dots, x_{n_i})\, \lambda(\mathrm{d}x_2) \dots \lambda(\mathrm{d}x_n) \\
&= \frac{1}{n_i!} \sum_{\tau,\sigma\in S_{n_i}} \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\psi^i_{\tau(1)}(x_1)\overline{\psi^i_{\sigma(1)}(x_1)} \prod_{k=2}^{n_i} \int \psi^i_{\tau(k)}(x_k)\overline{\psi^i_{\sigma(k)}(x_k)}\, \lambda(\mathrm{d}x_k) \\
&= \frac{1}{n_i!} \sum_{\sigma\in S_{n_i}} |\psi^i_{\sigma(1)}(x_1)|^2 \\
&= \frac{1}{n_i} \sum_{k=1}^{n_i} |\psi^i_k(x_1)|^2,
\end{aligned}$$

where S_n is the n-th symmetric group and $\mathrm{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_n$. By the same type of calculations, we can calculate the law of $X^i_2|X^i_1$, whose density with respect to λ is given by

$$\begin{aligned}
p_{2|X^i_1}(x_2) = \frac{p_2(X^i_1, x_2)}{p_1(X^i_1)} &= \frac{1}{(n_i-1)! \sum |\psi^i_j(X^i_1)|^2} \\
&\sum_{\sigma\in S_{n_i}} \left(|\psi^i_{\sigma(1)}(X^i_1)|^2 |\psi^i_{\sigma(2)}(x_2)|^2 - \psi^i_{\sigma(1)}(X^i_1)\overline{\psi^i_{\sigma(2)}}(X^i_1)\psi^i_{\sigma(2)}(x_2)\overline{\psi^i_{\sigma(1)}}(x_2)\right) \\
= \frac{1}{n_i - 1} &\left(\sum_{i=1}^{n_i} |\psi^i_j(x_2)|^2 - \Big| \sum_{j=1}^{n_i} \frac{\psi^i_j(X^i_1)}{\sqrt{\sum |\psi^i_j(X^i_1)|^2}} \overline{\psi^i_j}(x_2) \Big|^2 \right).
\end{aligned}$$

The previous formula can be generalized recursively, and has the advantage of giving a natural interpretation of the conditional densities. Indeed, we can write the conditional densities at each step in a way that makes the orthogonalization procedure appear. This is presented in the final algorithm, which was explicitly obtained in [44] (see also [34] for the proof). As in [44], we write $\mathbf{v}_i(x) = (\psi_1^i(x), \dots, \psi_{n_i}^i(x))^t$, where t stands for the transpose.

Algorithm 2 Simulation of determinantal projection point process

sample $X_{n_i}^i$ from the distribution with density $p_{n_i}(x) = \|\mathbf{v}_i(x)\|^2/n_i, \ x \in \Lambda$
$\mathbf{e}_1 \leftarrow \mathbf{v}_i(X_{n_i}^i)/\|\mathbf{v}(X_{n_i}^i)\|$
for $j = n_i - 1 \to 1$ **do**
  sample X_j^i from the distribution with density

$$p_j(x) = \frac{1}{j}\Big[\|\mathbf{v}_i(x)\|^2 - \sum_{k=1}^{n_i-j} |\mathbf{e}_k^* \mathbf{v}_i(x)|^2\Big]$$

  $\mathbf{w}_j \leftarrow \mathbf{v}_i(X_j) - \sum_{k=1}^{n_i-j} \left(\mathbf{e}_k^* \mathbf{v}_i(X_j^i)\right) \mathbf{e}_k, \quad \mathbf{e}_{n_i-j+1} \leftarrow \mathbf{w}_j/\|\mathbf{w}_j\|$
end for
return $(X_1^i, \dots, X_{n_i}^i)$

Then, Algorithm 2 yields a sample $(X_1^i, \dots, X_{n_i}^i,)$ which has a determinantal law with kernel $\tilde{K}^i(x,y) = \sum_{j=1}^{n_i} \psi_j^i(x)\overline{\psi_j^i(y)}$, $x, y \in \Lambda$.

Since $\mu_{\mathcal{K},\lambda}^{\alpha}$ is a superposition of $m+1$ independent copies of a determinantal point process with kernel $\mathcal{K}/(m+1)$, it suffices to apply Algorithm 2 $m+1$ times. The concatenation of the $m+1$ outputs of the algorithm therefore follows an α-determinantal law with kernel $\mathcal{K}$.

4.2 Perfect simulation of determinantal point processes

The basic idea developed in this section [1] is to apply coupling theory to simulate a finite state Markov chain which has a determinantal point process as equilibrium distribution. Here, the bound (3.19) enables us to apply a very general theory for the perfect simulation of point processes. The general framework and concepts have been developed in [70] (see also [40] and [41] for applications to point processes). Let us give here a brief overview of the concepts developed therein.

In the remainder of this section, we are given a compact set $\Lambda \subseteq E$, and turn our attention to the simulation of a determinantal point process with trace-class kernel $\mathcal{K}_\Lambda$. The main assumption to apply the algorithms of [37] and [41] is the following:

1. The ideas presented in this section are the result of a collaboration with L. Decreusefond and K. C. Low. We refer to Appendix C for details.

Hypothesis 6. *We assume that Hypothesis 4 is verified. Assume moreover that the following bound is satisfied:*

$$\sup_{x\in\Lambda} J(x,x) < \infty,$$

where J is the kernel of the operator $\mathcal{J}$ defined in (3.20).

When Hypothesis 6 is verified, the following bound holds for the Papangelou conditional intensity c:

$$c_\Lambda(x,\xi) \le J[\Lambda](x,x) \le \sup_{x\in\Lambda} J[\Lambda](x,x) = \sup_{x\in\Lambda} J(x,x) := H,$$

where we have used the bound (3.19), and the fact that $J[\Lambda](x,x) = J(x,x)$ for $x \in \Lambda$. In that case, the following algorithms were obtained in [41]:

Algorithm 3 Simulation of determinantal point process

Sample D_0 from PPP $(H|\Lambda|)$
$n \leftarrow 1/2$;
while TRUE **do**
  $D \leftarrow \text{BackwardExtend}(D, n)$;
  $[L, U] \leftarrow \text{Coupling}(D)$;
  if $L_0 == U_0$ **then**
    return L_0
  else
    $n \leftarrow n * 2$;
  end if
end while

Algorithm 4 BackwardExtend(D, n)

$j \leftarrow 0$;
$T(0) \leftarrow n/2$; $\{T(0) \leftarrow 0$ if $n = 1/2\}$
$\tilde{D}_{T(0)} \leftarrow D_{-n/2}$; $\{\tilde{D}_{T(0)} \leftarrow D_0$ if $n = 1/2\}$
while $T(j) \le n$ **do**
  $T(j+1) \leftarrow T(j) - \log(\text{Uniform}(0,1))/(H|\Lambda| + |\tilde{D}_{T(j)}|)$;
  if $\text{Uniform}(0,1) \le (H|\Lambda|)/(H|\Lambda| + |\tilde{D}_{T(j)}|)$ **then**
    $x \leftarrow$ uniform random point in $\Lambda \setminus \tilde{D}_{T(j)}$;
    $\tilde{D}_{T(j+1)} \leftarrow \tilde{D}_{T(j)} \cup x$;
  else
    $x \leftarrow$ uniform random point in $\tilde{D}_{T(j)}$;
    $\tilde{D}_{T(j+1)} \leftarrow \tilde{D}_{T(j)} \setminus x$;
  end if
  $j \leftarrow j + 1$;
end while
$D_{-t} \leftarrow \tilde{D}_t$ for all $t : n/2 < t \le n$
return D

Algorithm 5 Coupling(D)

$L_{-n} \leftarrow \emptyset$;
$U_{-n} \leftarrow D_{-n}$;
for $T_i \leftarrow$ each jump times $T_1 < T_2 < \cdots$ of D in $]-n:0]$ **do**
  if $D_{T_i} \leftarrow D_{T_{i-1}} \cup x$ **then**
    $u \leftarrow M_T$;
    $[L_{T_i}, U_{T_i}] \leftarrow \mathrm{AddBirth}(L_{T_{i-1}}, U_{T_{i-1}}, x, u)$;
  else
    $x \leftarrow D_{T_{i-1}} \setminus D_{T_i}$;
    $L_{T_i} \leftarrow L_{T_{i-1}} \setminus x$;
    $U_{T_i} \leftarrow U_{T_{i-1}} \setminus x$;
  end if
end for
return $[L, U]$

The output of Algorithm 10 is a perfect realization of the determinantal point process of kernel $\mathcal{K}_\Lambda$. Then, our main contribution is given by the following bound on the running time of the algorithm:

Proposition 4.2.1. *Assume that Hypothesis 6 holds. Then the process D_t which we introduced in Algorithm 10 is an $M/M/\infty$ queue defined on a compact space Λ with arrival rate $a = H|\Lambda|$ and service rate $s = 1$. Moreover, D_t is initiated at $t = t_0$, with $D_{t_0} = \mathrm{PPP}(H|\Lambda|)$, where $\mathrm{PPP}(\sigma)$ is a Poisson point process with intensity σ. Then we have*

$$\sup_{-t_0 \le t \le T} |D_t| = O\left(\max(x, H|\Lambda|)\right),$$

where $T \in \mathbb{R}$ and $x := \lim_{W\to\infty} \frac{|D_{t_0}^W|}{W}$.

If we fixed $x = H|\Lambda|$, we have that the stopping time of the algorithm is upper bounded by:

$$O(\log H|\Lambda|),$$

and heuristically, we are well below this upper bound.

4.3 Definition of the Ginibre point process

The Ginibre process, denoted by μ_{Gin} in the remainder of this paper, is defined as the determinantal process on $\mathbb{C}$ with integral kernel

$$K_{\mathrm{Gin}}(z_1, z_2) = \frac{1}{\pi} e^{z_1 \bar{z_2}} e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)}, \quad z_1, z_2 \in \mathbb{C}, \tag{4.3}$$

with respect to $\lambda := \mathrm{d}\ell(z)$, the Lebesgue measure on $\mathbb{C}$ (i.e. $\mathrm{d}\ell(z) = \mathrm{d}x\,\mathrm{d}y$, when $z = x + iy$). It can be naturally decomposed as:

$$K_{\mathrm{Gin}}(z_1, z_2) = \sum_{n\ge 0} \phi_n(z_1)\overline{\phi_n(z_2)}, \quad z_1, z_2 \in \mathbb{C},$$

where $\phi_n(z) := \frac{1}{\sqrt{\pi n!}} e^{-\frac{1}{2}|z|^2} z^n$, for $n \in \mathbb{N}$ and $z \in \mathbb{C}$. It can be easily verified that $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal family of $\mathrm{L}^2(\mathbb{C}, \mathrm{d}\ell)$. In fact, $(\phi_n)_{n\in\mathbb{N}}$ is a dense subset of $\mathrm{L}^2(\mathbb{C}, \mathrm{d}\ell)$. The Ginibre process μ verifies the following basic properties:

Proposition 4.3.1. *The Ginibre process μ_{Gin}, i.e. the determinantal process with kernel K_{Gin} satisfies the following:*

- *μ_{Gin} is ergodic with respect to the translations on the plane.*
- *μ_{Gin} is isotropic.*
- *$\mu_{\mathrm{Gin}}(\mathbb{C}) = +\infty$ almost surely, i.e. the Ginibre point process has an infinite number of points almost surely.*

Proof. For $a \in \mathbb{C}$, note that $K_{\mathrm{Gin}}(z_1 - a, z_2 - a) = K_{\mathrm{Gin}}(z_1, z_2)e^{-\frac{1}{2}a(\bar{z_2}-\bar{z_1})+\frac{1}{2}\bar{a}(z_1-z_2)}$, for $z_1, z_2 \in \mathbb{C}$. Hence,

$$\rho(z_1 - a, \dots, z_n - a) = \det(K_{\mathrm{Gin}}(z_i - a, z_j - a))_{1\leq i,j\leq n} = \det(K_{\mathrm{Gin}}(z_i, z_j))_{1\leq i,j\leq n},$$

which means that μ_{Gin} is invariant with respect to translations. Ergodicity with respect to translations follows from [80, Theorem 7].

Moreover, for $\theta \in \mathbb{R}$, we have $K_{\mathrm{Gin}}(z_1e^{i\theta}, z_2e^{i\theta}) = K_{\mathrm{Gin}}(z_1, z_2)$, for $z_1, z_2 \in \mathbb{C}$ (here and in the remainder of the paper, $i := \sqrt{-1}$). Hence, isotropy follows directly by uniqueness of the determinantal measure μ_{Gin}.

We have that $\mathrm{Tr}K_{\mathrm{Gin}} = +\infty$, hence by a classical result (see e.g. Theorem 4 in [80]), the number of points in μ_{Gin} is almost surely infinite. □

Since μ_{Gin} has an infinite number of points almost surely, it is impossible to simulate it directly. Therefore, in the remainder of this paper, we are interested in modifying the kernel K_{Gin} in order to obtain versions of the Ginibre point process which we can simulate.

4.4 Efficient simulation of the Ginibre point process

4.4.1 Truncated Ginibre point process

The first idea is to consider the truncated Ginibre kernel, defined for $N \in \mathbb{N}_*$ by

$$K^N_{\mathrm{Gin}}(z_1, z_2) = \sum_{n=0}^{N-1} \phi_n(z_1)\overline{\phi_n(z_2)}, \quad z_1, z_2 \in \mathbb{C}, \tag{4.4}$$

which is in fact a truncation of the sum in (4.3). Additionally, we call μ^N_{Gin} the associated determinantal point process with intensity measure $\mathrm{d}\ell$. We remark that μ^N_{Gin} tends to μ_{Gin} weakly, when N goes to infinity. As it is a projection kernel of type (3.11), we have seen previously that μ^N_{Gin} has N points almost surely. μ^N_{Gin} is clearly not translation invariant anymore; however, it remains isotropic for the same reason that μ_{Gin} is. Physically, μ^N_{Gin} is the distribution of N polarized electrons in a perpendicular magnetic field, filling the N lowest Landau levels, as is remarked in [74]. As μ^N_{Gin} has N points almost surely, it is entirely characterized by its joint distribution p which is calculated in the following proposition.

Proposition 4.4.1. *Let μ^N_{Gin} be the point process with kernel given by (4.4). Then, μ^N_{Gin} has N points almost surely and its joint density p is given by*

$$p(z_1, \dots, z_N) = \frac{1}{\pi^N}\prod_{p=0}^{N}\frac{1}{p!}\, e^{-\sum_{p=1}^N |z_p|^2} \prod_{1\leq p<q\leq N} |z_p - z_q|^2, \tag{4.5}$$

for $z_1, \dots, z_N \in \mathbb{C}$.

Proof.

$$p(z_1,\dots,z_N)=\frac{1}{N!}\det\left(K_{\mathrm{Gin}}^N(z_i,\,z_j)\right)_{1\le i,j\le N},\quad z_1,\dots,z_N\in\mathbb{C},$$

and in this case p can be explicitly calculated. Indeed, note that

$$p(z_1,\dots,z_N)=\frac{1}{N!}A^N(z_1,\dots,z_N)A^N(z_1,\dots,z_N)^*,$$

where the matrix $A^N:=(A^N_{ph})_{1\le p,h\le N}$ is given by

$$A^N_{ph}:=\phi_{h-1}(z_p)$$

and $A^N(z_1,\dots,z_N)^*$ denotes the transpose conjugate of $A^N(z_1,\dots,z_N)$. Hence,

$$p(z_1,\dots,z_N)=\frac{1}{N!}|\det A^N(z_1,\dots,z_N)|^2.$$

We recognize a Vandermonde determinant

$$\det A^N(z_1,\dots,z_N)=\left(\prod_{p=0}^{N-1}\sqrt{\frac{1}{\pi p!}}\right)e^{-\frac{1}{2}\sum_{p=1}^N|z_p|^2}\prod_{1\le p<q\le N}(z_p-z_q),$$

which leads to the following joint density for the N points:

$$p(z_1,\dots,z_N)=\frac{1}{\pi^N}\prod_{p=0}^{N}\frac{1}{p!}\,e^{-\sum_{p=1}^N|z_p|^2}\prod_{1\le p<q\le N}|z_p-z_q|^2,\quad z_1,\dots,z_N\in\mathbb{C}.$$

□

It is also known that the radii (in the complex plane) of the points of μ^N_{Gin} have the same distribution as independent gamma random variables. More precisely, we can find in [42] the following result:

Proposition 4.4.2. *Let $\{X_1,\dots,X_N\}$ be the $N\in\mathbb{N}_*$ unordered points, distributed according to μ^N. Then, $\{|X_1|,\dots,|X_N|\}$ has the same distribution as $\{Y_1,\dots,Y_N\}$, where for $1\le i\le N$, $Y_i^2\sim\mathrm{gamma}(i,1)$, and the Y_i are independent.*

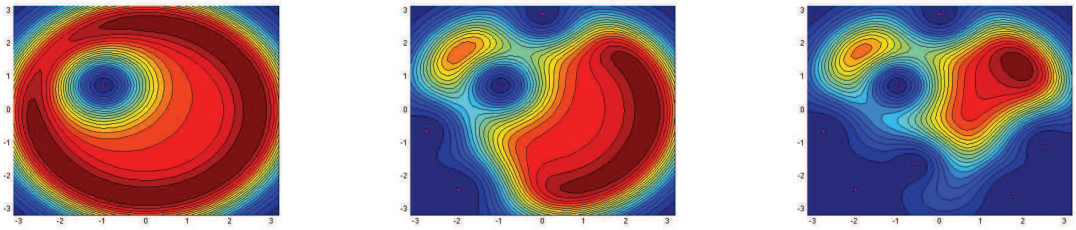
However, it should be noted that this does not yield a practical simulation technique, as the angles of $X_1,\dots,X_N$ are strongly correlated, and do not follow a known distribution.

We now move on to the problem of simulating a truncated Ginibre point process with kernel given by (4.4). Since μ^N_{Gin} has N points almost surely, there is no need to simulate the number of points. One only needs to simulate the positions of the N points. For this specific case, there is in fact a more natural way of simulating the Ginibre process. Indeed, it was proven in [30] that the eigenvalues of an $N\times N$ hermitian matrix with complex gaussian entries are distributed according to μ^N_{Gin}. More precisely, consider a matrix $N:=(N_{nm})_{1\le n,m\le N}$, such that for $1\le n,m\le N$,

$$N_{nm}=\frac{1}{\sqrt{2}}\left(N^1_{nm}+iN^2_{nm}\right),$$

where $N^1_{nm}, N^2_{nm} \sim \mathcal{N}(0,1)$, $1 \leq n, m \leq N$ are independent centered gaussian random variables. Then, the eigenvalues of N are distributed according to μ^N_{Gin}. This is by far the most efficient way of simulating the truncated Ginibre process.

We also remark that we could have applied the simulation technique of Section 4.1 in order to simulate the truncated Ginibre point process. However, the simulation procedure is much slower than calculating the eigenvalues of an $N \times N$ matrix. We still show the results of the algorithm of a realization of the resulting point process in the following. This allows proper visualization of the associated densities. We choose a window of size $a = 3$ and $N = 8$ in this example. We plot the densities p_i as color gradients before the simulation of the $(N-i)$-th point. The steps plotted in the following figure correspond to $i = 7$, $i = 4$, and $i = 1$ respectively (Algorithm 2 is used and is run from $i = N$ to $i = 1$). We also mark by red points the previously simulated points. Therefore, the point process obtained at the end of the algorithm consists of the red points in the third figure.



However, one runs into a practical problem when simulating the truncated Ginibre process: the support of its law is the whole of $\mathbb{C}^N$. Recall that the joint law of μ^N is known to be given by (4.5) which has support on $\mathbb{C}^N$. Moreover, projecting onto a compact subset randomizes the number of points in the point process. Therefore, this first method is only useful in applications where the point process need not be in a fixed compact subset of E.

4.4.2 Ginibre point process on a compact subset

We now consider more specifically the projection of the Ginibre process onto $\mathcal{B}_R$, and thus we consider the projection kernel $\mathcal{K}_{\mathrm{Gin},R} := P_{\mathcal{B}_R}\mathcal{K}_{\mathrm{Gin}}P_{\mathcal{B}_R}$ of the integral operator $\mathcal{K}_{\mathrm{Gin}}$ onto $\mathrm{L}^2(\mathcal{B}_R, \mathrm{d}\ell)$, where $\mathcal{B}_R := \overline{\mathcal{B}(0,R)}$ is the closed ball of $\mathbb{C}$ of radius $R \geq 0$ with center 0. In this specific case, the kernel of the operator $\mathcal{K}_{\mathrm{Gin},R}$ takes the form:

$$K_{\mathrm{Gin},R}(z_1, z_2) = \sum_{n \geq 0} \lambda^R_n \phi^R_n(z_1)\overline{\phi^R_n(z_2)}, \tag{4.6}$$

where $\phi^R_n(z) := Z^{-1}_{R,n}\phi_n(z)1_{z\in\mathcal{B}_R}$, $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $Z^{-1}_{R,n} \in \mathbb{R}$ is a constant depending only on R and n. This result does not hold in general, but is due to the fact that $(\phi^R_n(\cdot))_{n\geq 0}$ is

still an orthonormal family of $\mathrm{L}^2(\mathcal{B}_R, \mathrm{d}z)$. Indeed, for $m, n \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathcal{B}_R} \phi_n^R(z)\overline{\phi_m^R(z)}\,\mathrm{d}\ell(z) &= Z_{R,n}^{-2}\left(\frac{1}{\sqrt{n!m!}}\int_0^R r^{n+m+1}e^{-r^2}\,\mathrm{d}r\right)\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{i(n-m)\theta}\,\mathrm{d}\theta\right)\\ &= Z_{R,n}^{-2}1_{n=m}\left(\frac{2}{n!}\int_0^R r^{2n+1}e^{-r^2}\,\mathrm{d}r\right)\\ &= Z_{R,n}^{-2}1_{n=m}\frac{\gamma(n+1,R^2)}{n!},\end{aligned}$$

where γ is the lower incomplete Gamma function defined as

$$\gamma(z,a) := \int_0^a e^{-t}t^{z-1}\,\mathrm{d}t,$$

for $z \in \mathbb{C}$ and $a \geq 0$. Hence, in the following, we shall take $Z_{R,n} := \sqrt{\frac{\gamma(n+1,R^2)}{n!}}$. Therefore, the associated eigenvalues are

$$\lambda_n^R := \int_{\mathcal{B}_R} |\phi_n(z)|^2\,\mathrm{d}\ell(z) = Z_{R,n}^2 = \frac{\gamma(n+1,R^2)}{n!}.$$

As is expected, $0 \leq \lambda_n^R \leq 1$ for any $n \in \mathbb{N}, R \geq 0$, and $\lambda_n^R \xrightarrow[R\to\infty]{} 1$ for any $n \in \mathbb{N}$.

Now that we have specified the eigenvectors and associated eigenvalues, the simulation of the Ginibre process on a compact is that of the determinantal point process with kernel given by (4.6). Therefore, Algorithm 2 fully applies. The time-consuming step of the algorithm is the simulation of the Bernoulli random variables. Recall that the cumulative distribution function of $T = \sup\{n \in \mathbb{N}_* \,/\, B_n = 1\}$ is given by (4.2) which in our case is equal to

$$F(m) = \sum_{n\leq m} \frac{\gamma(n+1,R^2)}{n!}\prod_{i=n+1}^{\infty}\frac{\Gamma(i+1,R^2)}{i!},$$

for $m \in \mathbb{N}_*$.

We remark that we can not simulate the Ginibre point process restricted to a compact in the same way as in the previous subsection. Indeed, taking a $N \times N$ matrix with complex gaussian entries, and conditioning on the points being in $\mathcal{B}_R$ yields a determinantal point process with kernel,

$$K_{\mathrm{Gin},R}(z_1,z_2) = \sum_{n=0}^{N-1}\lambda_n^R\phi_n^R(z_1)\overline{\phi_n^R(z_2)},$$

which is not our target point process, as the sum is truncated at N. Therefore, the method developed in the previous subsection does not apply here. Hence, the algorithm is twofold, and the first step goes as follows:

Remark 3. *The series $\prod_{i\geq n}\frac{\Gamma(i+1,R^2)}{i!}$, for $n \in \mathbb{N}_*$ converges since it is equal to $\prod_{i\geq n}(1-\lambda_i^R)$ which is convergent. Indeed, $\sum_{i\geq 0}\lambda_i^R < \infty$ since the considered operator is locally trace-class.*

Algorithm 6 Simulation of the Ginibre process on a compact subset (Step 1)

evaluate numerically $R \leftarrow \prod_{i\geq 1} \frac{\Gamma(i+1,R^2)}{i!}$, for example by calculating $e^{\sum_{i=1}^{N} \ln(\frac{\Gamma(i+1,R^2)}{i!})}$, where N is chosen such that $\ln(\frac{\Gamma(N+1,R^2)}{N!}) < \epsilon$, $\epsilon > 0$ given by the user.
sample $U \leftarrow \mathcal{U}([0,1])$ according to a uniform distribution on $[0,1]$.
$m \leftarrow 0$
while $U < R$ **do**
  $m \leftarrow m+1$
  $R \leftarrow \frac{m!\gamma(m+1,R^2)}{\gamma(m,R^2)\Gamma(m+1,R^2)} R$
end while
for $i = 0 \rightarrow m-1$ **do**
  $B_i \leftarrow \mathrm{Be}(\frac{\gamma(i+1,R^2)}{i!})$, where here $\mathrm{Be}(\lambda)$ is an independent drawing of a Bernoulli random variable of parameter $\lambda \in [0,\, 1]$
end for
if $m > 0$ **then**
  return $\{B_0, \ldots, B_{m-1}, 1\}$
end if
if $m = 0$ **then**
  return $\{1\}$
end if

We write $\{B_0, \ldots, B_{m-1}, 1\}$ for the value returned by the previous algorithm, with the convention that $\{B_0, \ldots, B_{m-1}, 1\} = \{1\}$ if $m = 0$. Then by Theorem 4.1.1, the law of the Ginibre point process on a compact is the same as that of the determinantal point process of kernel

$$K_{\mathrm{Gin},R}(z_1, z_2) = \sum_{n=0}^{N-1} B_n \phi_n^R(z_1) \overline{\phi_n^R(z_2)}, \quad z_1, z_2 \in \mathbb{C}.$$

Now, we move onto the second part of the algorithm, which is this time straightforward as it suffices to follow Section 4.1 closely.

Algorithm 7 Simulation of the Ginibre process on a compact subset (Step 2)

define $\phi_k(z) := \frac{1}{\pi\gamma(k+1,R^2)} e^{-\frac{1}{2}|z|^2} z^k$, for $z \in \mathcal{B}_R$ and $0 \leq k \leq m$.
define $\mathbf{v}(z) := (\phi_{i_0}(z), \dots, \phi_{i_k}(z), \phi_m(z))$, for $z \in \mathcal{B}_R$, and where $\{i_0, \dots, i_k\} = \{0 \leq i \leq m-1 \ : \ B_i = 1\}$
$N \leftarrow k+2$
sample X_N from the distribution with density $p_N(x) = \|\mathbf{v}(x)\|^2/N, \ \ x \in \Lambda$
$\mathbf{e}_1 \leftarrow \mathbf{v}(X_N)/\|\mathbf{v}(X_N)\|$
for $i = N-1 \to 1$ **do**
  sample X_i from the distribution with density

$$p_i(x) = \frac{1}{i}\Big[\|\mathbf{v}(x)\|^2 - \sum_{j=1}^{N-i} |\mathbf{e}_j^* \mathbf{v}(x)|^2\Big]$$

  $\mathbf{w}_i \leftarrow \mathbf{v}(X_i) - \sum_{j=1}^{N-i} \left(\mathbf{e}_j^* \mathbf{v}(X_i)\right) \mathbf{e}_j, \quad \mathbf{e}_{N-i+1} \leftarrow \mathbf{w}_i/\|\mathbf{w}_i\|$
end for
return $(X_1, \dots, X_N)$

We end this subsection by mentioning the difficulties arising in the simulation under the density p_i, $1 \leq i \leq N-1$. As is remarked in [44], in the general case, we have no choice but to simulate by rejection sampling and the Ginibre point process is no different (except in the case $i = N-1$ where p_i is the density of a gaussian random variable). Therefore in practice, we draw a uniform random variable u on $\mathcal{B}_R$ and choose $p_i(u)/\sup_{y\in\mathcal{B}_R} p_i(y)$. Note that the authors in [44] give a closed form bound on p_i which is given by

$$p_i(x) \leq \frac{1}{i} \min_{i+1\leq k\leq N} \left(K_{\mathrm{Gin}}^N(x,x) - \frac{|K_{\mathrm{Gin}}^N(x,X_k)|^2}{K_{\mathrm{Gin}}^N(X_k,X_k)} \right), \tag{4.7}$$

where $X_{i+1}, \dots, X_N$ is the result of the simulation procedure up to step i. In practice however, the error made in the previous inequality is not worth the gain made by not evaluating $\sup_{y\in\mathcal{B}_a} p_i(y)$. Therefore, in our simulations, we have chosen not to use (4.7).

4.4.3 Truncated Ginibre process on a compact subset

In this subsection, we begin by studying the truncated Ginibre point process on a compact subset, and specifically discuss the optimal choice of the compact subset onto which we project. We begin by studying the general projection of the truncated Ginibre process onto a centered ball of radius $R \geq 0$ which is again a determinantal point process whose law can be obtained. To that end, we wish to study $K_R^N := P_{\mathcal{B}_R} K^N P_{\mathcal{B}_R}$ of the integral operator K onto $\mathrm{L}^2(\mathcal{B}_R, \mathrm{d}\ell)$. The associated kernel is given by

$$K_{\mathrm{Gin},R}^N(z_1,z_2) = \sum_{n=0}^{N-1} \lambda_n^R \phi_n^R(z_1)\overline{\phi_n^R(z_2)}, \tag{4.8}$$

for $z_1, z_2 \in \mathcal{B}_R$. The question of the Janossy densities of the associated determinantal process is not as trivial as the non-projected one. Indeed, $\mu^N_{\mathrm{Gin},R}$, the associated determinantal point process, does not have N points almost surely. However, it is known that it has less than N points almost surely (see e.g. [80]). Therefore, it suffices to calculate the Janossy densities $j^0_R, \dots, j^N_R$ to characterize the law of $\mu^N_{\mathrm{Gin},R}$. These are given by the following proposition:

Proposition 4.4.3. *The point process $\mu^N_{\mathrm{Gin},R}$ with kernel given by (4.8) has less than N points almost surely, and its Janossy densities are given by*

$$j^k_{\mathrm{Gin},R}(z_1,\dots,z_k) = \frac{1}{\pi^k}\prod_{p=0}^{k-1}\frac{1}{p!}e^{-\sum_{p=1}^k |z_p|^2}\prod_{1\le i<j\le k}|z_i-z_j|^2 \sum_{\{i_1,\dots,i_k\}\subset\{1,\dots,N\}} |s_{\lambda(i_1,\dots,i_k)}(z_1,\dots,z_k)|^2,$$

for $0 \le k \le N$ and $z_1, \dots, z_k \in \mathcal{B}_R$. Here, s_λ is the Schur polynomial defined e.g. in [33].

Proof. By formula (3.10), the operator $\mathcal{J}^N_{\mathrm{Gin}}[\mathcal{B}_R]$ associated to μ^N_{Gin} has a kernel that can be decomposed as:

$$J^N_{\mathrm{Gin}}[\mathcal{B}_R](z_1,z_2) = \sum_{n=0}^{N-1}\frac{\gamma(n+1,R^2)}{\Gamma(n+1,R^2)}\,\phi^R_n(z_1)\overline{\phi^R_n(z_2)}, \quad z_1,z_2\in\mathcal{B}_R,$$

where Γ is the upper incomplete Gamma function defined as

$$\Gamma(z,a) := \int_a^\infty e^{-t}t^{z-1}\,\mathrm{d}t,$$

for $z \in \mathbb{C}$ and $a \ge 0$, which by definition verifies $\gamma(\cdot,a)+\Gamma(\cdot,a) = \Gamma(\cdot)$ for all $a \ge 0$ ($\Gamma(\cdot)$ is the usual Gamma function). Here, we note that $j^N_{\mathrm{Gin},R}$ can be calculated as previously as the associated determinant is again a Vandermonde determinant. More precisely, we obtain

$$\det\left(J^N_{\mathrm{Gin}}[\mathcal{B}_R](z_i,z_j)\right)_{1\le i,j\le N} = \frac{1}{\pi^N}\prod_{p=0}^{N-1}\frac{1}{\Gamma(p+1,R^2)}\,e^{-\sum_{p=1}^N|z_p|^2}\prod_{1\le p<q\le N}|z_p-z_q|^2,$$

for $z_1, \dots, z_N \in \mathcal{B}_R$. Moreover, the void probability, i.e. the probability of having no point in $\mathcal{B}_R$, is equal to

$$\mathrm{Det}\left(\mathrm{I}-K^N_{\mathrm{Gin},R}\right) = \prod_{n=0}^{N-1}(1-\lambda^R_n) = \prod_{n=0}^{N-1}\frac{\Gamma(n+1,R^2)}{n!}. \tag{4.9}$$

Hence, we obtain the following expression for the N-th Janossy density:

$$j^N_{\mathrm{Gin},R}(z_1,\dots,z_N) = \frac{1}{\pi^N}\prod_{p=0}^{N-1}\frac{1}{p!}\,e^{-\sum_{p=1}^N|z_p|^2}\prod_{1\le p<q\le N}|z_p-z_q|^2,$$

for $z_1, \dots, z_N \in \mathcal{B}_R$. Now, if we take $k < N$, we have again

$$J^N_{\mathrm{Gin}}[\mathcal{B}_R](z_1,\dots,z_k) = A^N(z_1,\dots,z_k)A^N(z_1,\dots,z_k)^*,$$

where this time, $A^N(z_1,\dots,z_k)$ is a rectangular $k\times N$ matrix. Hence, by application of the Cauchy-Binet formula:

$$\det J^N_{\mathrm{Gin}}[\mathcal{B}_R](z_1,\dots,z_k)=\sum_{\{i_1,\dots,i_k\}\subset\{1,\dots,N\}}|\det A^{i_1,\dots,i_k}(z_1,\dots,z_k)|^2,$$

where we have for $1\le p,h\le k$,

$$A^{i_1,\dots,i_k}_{ph}(z_1,\dots,z_k):=\sqrt{\frac{\gamma(n+1,R^2)}{\Gamma(n+1,R^2)}}\phi^R_{i_h}(z_p),$$

which is a square matrix. We now consider fixed $\{i_1,\dots,i_k\}\subset\{1,\dots,N\}$ and wish to evaluate $|\det A^{i_1,\dots,i_k}(z_1,\dots,z_k)|^2$. In fact, we observe that

$$|\det A^{i_1,\dots,i_k}(z_1,\dots,z_k)|^2=\prod_{p=0}^{k-1}\frac{1}{\pi\Gamma(p+1,R^2)}e^{-\sum_{p=1}^k|z_p|^2}\left|V_{i_1,\dots,i_k}(z_1,\dots,z_k)\right|^2,$$

where

$$V_{i_1,\dots,i_k}(z_1,\dots,z_k):=\det\left(\left(z_h^{i_p}\right)_{1\le p,h\le k}\right)$$

is known in the literature as the generalized Vandermonde determinant. Here, $V_{1,\dots,k}(z_1,\dots,z_k)$ is the classical Vandermonde determinant, and in the general case, a certain number of rows from the matrix have been deleted. The generalized Vandermonde determinant is known to factorize into the classical Vandermonde determinant and what is defined to be a Schur polynomial. To be more precise,

$$V_{i_1,\dots,i_k}(z_1,\dots,z_k)=V_{1,\dots,k}(z_1,\dots,z_k)s_{\lambda(i_1,\dots,i_k)}(z_1,\dots,z_k),$$

where $\lambda(i_1,\dots,i_k):=(i_k-k+1,\dots,i_2-1,i_1)$, and s_λ is the Schur polynomial, which is known to be symmetric, and is a sum of monomials, see e.g. [33]. To summarize, we have

$$\det J^N_{\mathrm{Gin}}[\mathcal{B}](z_1,\dots,z_k)=\left(\prod_{p=0}^{k-1}\frac{1}{\pi\Gamma(p+1,R^2)}\right)e^{-\sum_{p=1}^k|z_p|^2}\prod_{1\le i<j\le k}|z_i-z_j|^2\sum_{\{i_1,\dots,i_k\}\subset\{1,\dots,N\}}|s_{\lambda(i_1,\dots,i_k)}(z_1,\dots,z_k)|^2.$$

Then, by (4.9), we find

$$j^k_{\mathrm{Gin},R}(z_1,\dots,z_k)=\frac{1}{\pi^k}\prod_{p=0}^{k-1}\frac{1}{p!}e^{-\sum_{p=1}^k|z_p|^2}\prod_{1\le i<j\le k}|z_i-z_j|^2\sum_{\{i_1,\dots,i_k\}\subset\{1,\dots,N\}}|s_{\lambda(i_1,\dots,i_k)}(z_1,\dots,z_k)|^2,\quad(4.10)$$

for $z_1,\dots,z_k\in\mathcal{B}_R$. □

Next, we wish to determine the optimal $R\ge 0$ onto which we project the truncated Ginibre process. In regards to this question, we recall that the particle density ρ_1 of the general Ginibre process is constant, and

$$\rho_1(z)=K_{\mathrm{Gin}}(z,z)=\frac{1}{\pi},$$

for $z \in \mathbb{C}$. However, the particle density of the truncated Ginibre process is not constant. If we denote by ρ_n^N the n-th correlation function of μ_{Gin}^N, then we have

$$\rho_1^N(z) = \frac{1}{\pi} e^{-|z|^2} \sum_{k=0}^{N-1} \frac{|z|^{2k}}{k!},$$

for $z \in \mathbb{C}$. As can be checked easily, we have $\int_{\mathbb{C}} \rho_1^N(z)\, \mathrm{d}z = N$ as well as

$$\rho_1^N(z) \leq \frac{1}{\pi}, \quad \forall z \in \mathbb{C}, \tag{4.11}$$

and in fact it is known that $\rho_1^N(\sqrt{N}z) \xrightarrow[N\to\infty]{} \frac{1}{\pi} 1_{|z|\leq 1}$, which is known as the circular law in stochastic matrix theory. It therefore appears that it is optimal to project onto $\mathcal{B}_{\sqrt{N}}$. We wish to get more precise results on the error we are making by truncating the point process to $\mathcal{B}_R$. To that end, we recall the following bounds on ρ_1^N which were obtained in [30]. We recall their proof for convenience, as our bounds differ slightly from the ones obtained there.

Proposition 4.4.4. *For $|z|^2 < N+1$, we have*

$$\frac{1}{\pi} - \rho_1^N(z) \leq \frac{1}{\pi} e^{-|z|^2} \frac{|z|^{2N}}{N!} \frac{N+1}{N+1-|z|^2}.$$

For $|z|^2 \geq N-1$, we have

$$\rho_1^N(z) \leq \frac{1}{\pi} e^{-|z|^2} \frac{|z|^{2N}}{(N-1)!} \frac{1}{|z|^2 - N + 1}.$$

Proof. By using $\frac{(k+N)!}{N!} \geq (N+1)^k$, for $k, N \in \mathbb{N}$, we obtain for $|z|^2 < N+1$,

$$\begin{aligned}\rho_1^N(z) = \frac{1}{\pi} - \frac{1}{\pi} e^{-|z|^2} \sum_{k=N}^{\infty} \frac{|z|^{2k}}{k!} &= \frac{1}{\pi} - \frac{1}{\pi} e^{-|z|^2} \frac{|z|^{2N}}{N!} \sum_{k=0}^{\infty} \frac{|z|^{2k} N!}{(k+N)!} \\ &\geq \frac{1}{\pi} - \frac{1}{\pi} e^{-|z|^2} \frac{|z|^{2N}}{N!} \frac{1}{1 - \frac{|z|^2}{N+1}}.\end{aligned}$$

The proof of the second inequality is along the same lines, except that we use $\frac{N!}{(N-k)!} \leq N^k$, for $N \in \mathbb{N}$ and $0 \leq k \leq N$. Specifically, we have,

$$\begin{aligned}\rho_1^N(z) = \frac{1}{\pi} e^{-|z|^2} \sum_{k=0}^{N-1} \frac{|z|^{2(N-1-k)}}{(N-1-k)!} &= \frac{1}{\pi} e^{-|z|^2} \frac{|z|^{2(N-1)}}{(N-1)!} \sum_{k=0}^{N-1} \frac{|z|^{-2k}(N-1)!}{(N-1-k)!} \\ &\leq \frac{1}{\pi} e^{-|z|^2} \frac{|z|^{2N}}{(N-1)!} \frac{1}{|z|^2 - (N-1)}.\end{aligned}$$

□

As was noticed in [30], if we set $|z| = \sqrt{N} + u$, for $-1 \leq u \leq 1$, both of the right hand sides of the inequalities in Proposition 4.4.4 tend to

$$\frac{1}{2\sqrt{2} u \pi^{3/2}} e^{-2u^2},$$

as N tends to infinity. This is obtained by standard calculations involving in particular the Stirling formula. That is to say, for $|z| \leq \sqrt{N}$, and $|z| = \sqrt{N} - u$, we write $\rho_1^N(z) := \rho_1^N(|z|)$ and hence,

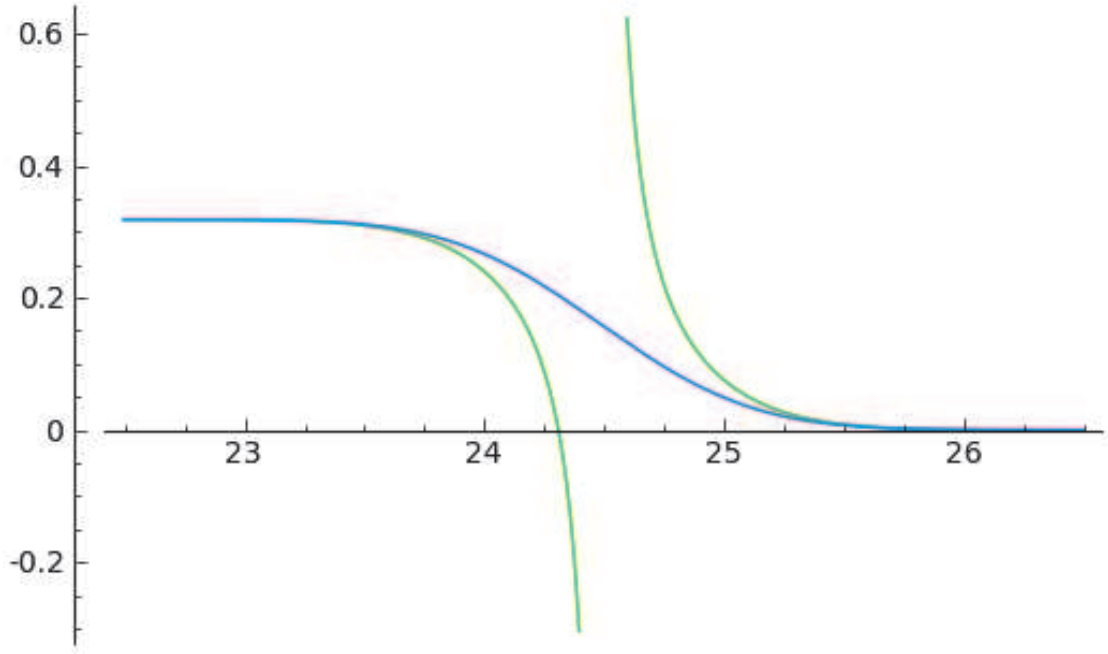
$$\rho_1^N(\sqrt{N} - u) \geq \frac{1}{\pi} - \frac{1}{2\sqrt{2}u\pi^{3/2}} e^{-2u^2}, \tag{4.12}$$

as well as for $|z| \geq \sqrt{N}$, and $|z| = \sqrt{N} + u$

$$\rho_1^N(\sqrt{N} + u) \leq \frac{1}{2\sqrt{2}u\pi^{3/2}} e^{-2u^2}, \tag{4.13}$$

as N tends to infinity. These bounds exhibit the sharp fall of the particle density around $|z| = \sqrt{N}$.

Figure 4.1: $\rho_1^N(|z|)$ for $N = 600$ and $|z|$ around $\sqrt{N}$ (blue). Upper and lower bounds obtained in (4.12) and (4.13) (green).



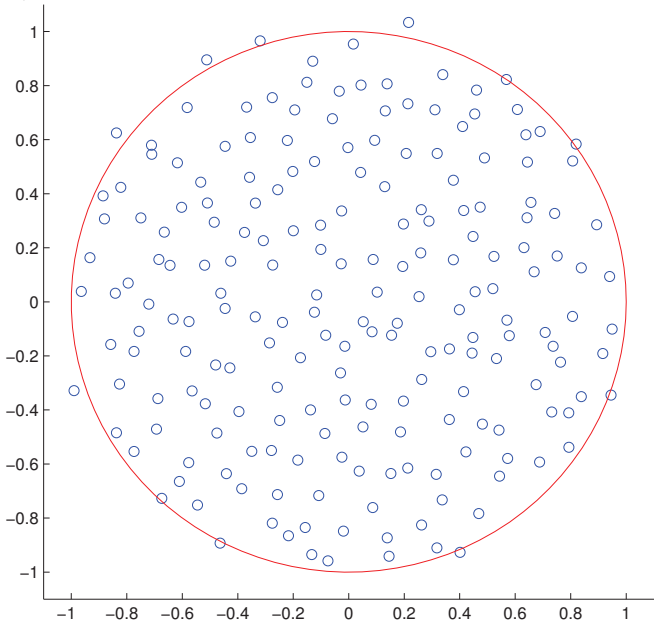
Even though the kernel ought to be projected onto $\mathcal{B}_{\sqrt{N}}$, this projection randomizes the number of points in the point process, as is seen in Figure 4.2.

Therefore, our additional idea is to condition the number of points on being equal to N. Since the projection onto $\mathcal{B}_{\sqrt{N}}$ of the truncated Ginibre process takes the determinantal form (4.8), one can easily calculate the probability of all the points falling in $\mathcal{B}_{\sqrt{N}}$. Indeed, we have that

$$\mathbb{P}_{\mu^N}(\xi_{\mathcal{B}^c_{\sqrt{N}}} = \emptyset) = \prod_{n=0}^{N-1} \lambda_n^N = \prod_{n=0}^{N-1} \frac{\gamma(n+1, N)}{n!}. \tag{4.14}$$

It can be shown that this probability tends to 0 as N tends to infinity. That is, if we are required to simulate the Ginibre process on a compact conditionally on it having N points,

Figure 4.2: A realization of μ^N for $N = 200$ (blue circles) renormalized to fit in the circle of radius 1 (in red)



the conditioning requires more and more computation time as N tends to infinity.

However, we are not forced to simulate the conditioning on there being N points. Instead, we introduce a new kernel, as well as the associated point process. We set

$$\tilde{K}^N_{\mathrm{Gin}}(z_1, z_2) = \sum_{n=0}^{N-1} \phi^N_n(z_1)\overline{\phi^N_n(z_2)}, \quad z_1, z_2 \in \mathcal{B}_R, \tag{4.15}$$

and where ϕ^N_n corresponds to the function ϕ_n restricted to the compact $\mathcal{B}_{\sqrt{N}}$ (after renormalization). We emphasize that $\tilde{\mu}^N_{\mathrm{Gin}}|_{\mathcal{B}_{\sqrt{N}}}$ is in fact $\mu^N_{\mathrm{Gin}}|_{\mathcal{B}_{\sqrt{N}}}$ conditioned on there being N points in the compact $\mathcal{B}_{\sqrt{N}}$, this result being due to Theorem 4.1.1. Moreover, the determinantal point process associated with this kernel benefits from the efficient simulations techniques developed in the previous subsection. Here, the fact that we can explicit the projection kernel associated with the conditioning is what ensures the efficiency of the simulation.

Let us start by proving that $\tilde{\mu}^N_{\mathrm{Gin}}$, the associated determinantal process with kernel $\tilde{K}^N_{\mathrm{Gin}}$, converges to μ weakly as N tends to infinity. This is a consequence of Proposition 3.3.4, as is proved in the following:

Theorem 4.4.1. *We have that $\tilde{K}^N_{\mathrm{Gin}}$ converges uniformly on compact subsets to K_{Gin} as N tends to infinity. As a consequence, the associated determinantal measures converge weakly to the determinantal point process of kernel K_{Gin}.*

Proof. Take a compact subset A of $\mathbb{C}$, and write $|A| := \sup\{|z|,\ z \in A\}$. Then, for

$z_1, z_2 \in A$,

$$|K_{\mathrm{Gin}}(z_1,z_2) - \tilde{K}^N_{\mathrm{Gin}}(z_1,z_2)| \leq \sum_{n=0}^{N-1} \frac{1}{\pi}|A|^{2n}\Big|\frac{1}{\gamma(n+1,N)} - \frac{1}{n!}\Big|1_{\{(z_1,z_2)\in(\mathcal{B}_N)^2\}} + \sum_{n=N}^{\infty}\frac{1}{\pi n!}|A|^{2n} + \sum_{n=0}^{N-1}\frac{1}{\pi n!}|A|^{2n}1_{\{(z_1,z_2)\notin(\mathcal{B}_N)^2\}}.$$

The second term tends to zero as the remainder of a convergent series, and the third term also tends to zero by dominated convergence. Concerning the first term, we need sightly more precise arguments. Let us start by rewriting it as

$$\sum_{n=0}^{\infty}\frac{1}{\pi}|A|^{2n}\frac{1}{\gamma(n+1,N)}1_{\{(z_1,z_2)\in(\mathcal{B}_N)^2\}}1_{\{n\leq N-1\}} - \sum_{n=0}^{N-1}\frac{1}{\pi n!}|A|^{2n}1_{\{(z_1,z_2)\in(\mathcal{B}_N)^2\}}, \quad (4.16)$$

and noticing that $\gamma(n+1,N) \to n!$ as N tends to infinity. Therefore, in order to conclude, we wish to exhibit a summable bound. To this end, we write

$$\begin{aligned}\frac{1}{\gamma(n+1,N)}1_{\{n\leq N-1\}} &\leq \frac{1}{\gamma(n+1,n+1)}\\ &= \frac{1}{n!\,\mathbb{P}(\sum_{k=1}^{n+1}X_k \leq n+1)}\\ &\sim_{n\to\infty} \frac{2}{n!}\end{aligned}$$

where $X_1,\ldots,X_n$ are independent exponential random variables of parameter 1. In the previous calculations, we have used the fact that $\frac{\gamma(a,R)}{\Gamma(a)}$ is the cumulative distribution function of a $\Gamma(a,R)$ random variable, $a>0$, and $R\geq 0$. The last line results from the application of the central limit theorem to $X_1,\ldots,X_n$. Hence,

$$\sum_{n=0}^{\infty}\frac{1}{\pi}|A|^{2n}\frac{1}{\gamma(n+1,N)}1_{\{(z_1,z_2)\in(\mathcal{B}_N)^2\}}1_{\{n\leq N-1\}} \leq \sum_{n=0}^{\infty}\frac{1}{\pi\gamma(n+1,n+1)}|A|^{2n} < \infty,$$

which means that by Lebesgue's dominated convergence theorem, (4.16) tends to zero as N tends to infinity. Therefore, $|K_{\mathrm{Gin}}(z_1,z_2) - \tilde{K}^N_{\mathrm{Gin}}(z_1,z_2)| \xrightarrow[N\to\infty]{} 0$ for $z_1,z_2\in A$. Hence, Proposition 3.3.4 allows us to conclude that $\tilde{\mu}^N_{\mathrm{Gin}} \xrightarrow[N\to\infty]{\text{weakly}} \mu_{\mathrm{Gin}}$. □

We now return to the problem of simulating the determinantal point process with kernel given by (4.15). As it is a projection process, it is efficiently simulated according to the basic algorithm described in Section 4.1. On the other hand, the time-consuming step of generating the Bernoulli random variables is not necessary anymore, as we are working conditionally on there being N points. Lastly, the method described in this section yields a determinantal point process on $\mathcal{B}_{\sqrt{N}}$. As before, in order to simulate on $\mathcal{B}_a$, we need to apply a homothetic transformation to the N points, which translates to a homothety on the eigenvectors. To sum up, the simulation algorithm of the truncated Ginibre process on a centered ball of radius $a\geq 0$ is as follows:

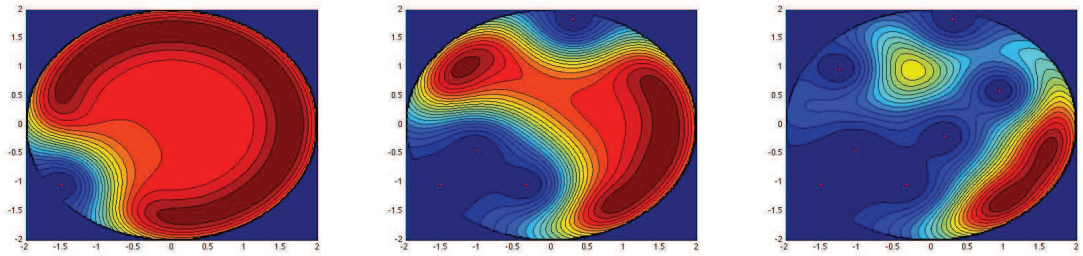
Algorithm 8 Simulation of the truncated Ginibre process on a compact

define $\phi_k(z) = \frac{N}{\pi a^2 \gamma(k+1,N)} e^{-\frac{N}{2a^2}|z|^2} (\frac{Nz}{a^2})^k$, for $z \in \mathcal{B}_N$ and $0 \leq k \leq N-1$.
define $\mathbf{v}(z) := (\phi_0(z), \dots, \phi_{N-1}(z))$, for $z \in \mathcal{B}_N$.
sample X_N from the distribution with density $p_N(x) = \|\mathbf{v}(x)\|^2/N, \ \ x \in \Lambda$
set $\mathbf{e}_1 = \mathbf{v}(X_N)/\|\mathbf{v}(X_N)\|$
for $i = N-1 \to 1$ **do**
  sample X_i from the distribution with density

$$p_i(x) = \frac{1}{i}\Big[\|\mathbf{v}(x)\|^2 - \sum_{j=1}^{N-i} |\mathbf{e}_j^* \mathbf{v}(x)|^2\Big]$$

  set $\mathbf{w}_i = \mathbf{v}(X_i) - \sum_{j=1}^{N-i} \Big(\mathbf{e}_j^* \mathbf{v}(X_i)\Big) \mathbf{e}_j, \quad \mathbf{e}_{N-i+1} = \mathbf{w}_i/\|\mathbf{w}_i\|$
end for
return $(X_1, \dots, X_N)$

The resulting process is a determinantal point process of kernel (4.15). Its support is on the compact $\mathcal{B}_a$ and has N points almost surely. We now give a brief example of the results of the algorithm applied for $a = 2$ and $N = 9$ at steps $i = 8$, $i = 5$, and $i = 2$ respectively. We have plotted the densities used for the simulation of the next point. We note here that the density is now supported on $\mathcal{B}_a$, whereas before the density was decreasing to zero outside of $\mathcal{B}_a$.



This determinantal point process presents the advantage of being easy to use in simulations, as well as having N points almost surely. Moreover, Theorem 4.4.1 proves its convergence to the Ginibre point process as N tends to infinity.

4.5 Summary

In this chapter, we have studied and discussed two very general simulation techniques, the first one which traces back to [34] was detailed in Section 4.1. The second one, which was presented in Section 4.2, uses a general simulation technique for point processes applied to the case of determinantal point processes.

These two simulation methods are conceptually quite different and are therefore difficult to compare. To be more precise, in Section 4.1, there are two time-consuming steps: the simulation of the Bernoulli random variables and the simulation under the density p_i for

which we are a priori required to proceed by rejection sampling. This requires an evaluation of the supremum of p_i on a grid which can be unboundedly big. On the other hand, in Section 4.2, we avoid the previous problem by only evaluating elaborate functionals (in our case, the Papangelou conditional intensity c) on a specific configuration, and not on the whole grid. However, the time necessary to reach equilibrium can be quite long, which is the main drawback of this algorithm.

In the future, we think comparing both approaches could provide quantitative answers to the questions raised in the previous paragraph. We gave a first answer in Proposition 4.2.1, but a comparison with the algorithm of Section 4.1 is quantitatively difficult since it seems to intrinsically depend on the kernel $\mathcal{K}$ of the underlying determinantal point process.

Lastly, we studied more specifically the Ginibre point process, which is a specific determinantal point process appearing in stochastic matrix theory. In this specific case, we realized that the usual general algorithms were not adapted. Therefore, we modified slightly the considered kernel in order to optimize execution time. This modification of the kernel yields an interesting model which could be used in applications necessitating a point process with repulsion. We have discussed in Section 4.3 the advantages of the Ginibre point process with regards to applications.

Chapter 5

Stochastic analysis on the configuration space

In Section 5.1, we give the necessary background on Dirichlet form theory. The main references for this section are [21, 27, 49], see also [90]. Then, in Section 5.2, we move more specifically to the configuration space, and define a gradient which will be found to satisfy an integration by parts. In Section 5.3, we move onto the specific case of determinantal point processes, and prove the existence of associated diffusions. To conclude, in Section 5.4, we generalize most results to any point process satisfying Hypothesis 4.

5.1 Introduction to Dirichlet forms

In this section, we are given an operator $\mathcal{E}$ defined on $\mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E})$ and with values in $\mathbb{R}$. Here as usual $\mathcal{X}$ is the configuration space, and μ is a measure on $(\mathcal{X}, \mathcal{F})$. We call $\mathrm{Dom}(\mathcal{E}) \subset \mathrm{L}^2(\mathcal{X}, \mu)$ the domain of $\mathcal{E}$. Here, $\mathrm{Dom}(\mathcal{E})$ is a dense linear subset of $\mathrm{L}^2(\mathcal{X}, \mu)$. We also assume that $\mathcal{E}$ is a nonnegative definite symmetric bilinear form, i.e. $\mathcal{E}(f, f) \geq 0$, for all $f \in \mathrm{Dom}(\mathcal{E})$.

5.1.1 Symmetric forms

For $\alpha > 0$, define a new symmetric form on $\mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E})$, by

$$\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha \langle u, v \rangle_{\mathrm{L}^2(\mathcal{X}, \mu)}.$$

We then define the norm associated to $\mathcal{E}_\alpha$ by

$$\|u\|_\alpha := \sqrt{\mathcal{E}(u, u) + \alpha \|u\|^2_{\mathrm{L}^2(\mathcal{X}, \mu)}},$$

for $u \in \mathrm{Dom}(\mathcal{E}) \cap \mathrm{L}^2(\mathcal{X}, \mu)$. We say that $\mathcal{E}$ is closed if $\mathrm{Dom}(\mathcal{E})$ is complete with respect to the metric determined by $\mathcal{E}_\alpha$. In other words, $\mathcal{E}$ is closed if

$$(u_n)_{n \in \mathbb{N}} \in \mathrm{Dom}(\mathcal{E})^{\mathbb{N}},\ \mathcal{E}_1(u_n - u_m, u_n - u_m) \xrightarrow[n,m \to \infty]{} 0$$
$$\Longrightarrow \quad \exists u \in \mathrm{Dom}(\mathcal{E}),\ \mathcal{E}_1(u_n - u, u_n - u) \xrightarrow[n \to \infty]{} 0.$$

We say that $\mathcal{E}$ is closable if

$$(u_n)_{n\in\mathbb{N}} \in \mathrm{Dom}(\mathcal{E})^{\mathbb{N}},\ \mathcal{E}(u_n-u_m,u_n-u_m) \xrightarrow[n,m\to\infty]{} 0 \ \text{ and } \ (u_n,u_n) \xrightarrow[n\to\infty]{} 0 \\ \implies \quad \mathcal{E}(u_n,u_n) \xrightarrow[n\to\infty]{} 0. \quad (5.1)$$

When $\mathcal{E}$ is closable, we denote by $(\overline{\mathcal{E}}, \overline{\mathrm{Dom}(\mathcal{E})})$ the closure of $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ with respect to $\|\cdot\|_1$. This definition is adequate since a necessary and sufficient condition for a symmetric form to possess a closed extension is that the symmetric form is closable.

Definition 5 (Markovian form)**.** *We say that $\mathcal{E}$ is Markovian if*

$$\forall \epsilon > 0,\ \exists \phi_\epsilon : \mathbb{R} \to \mathbb{R},$$

such that

$$\begin{cases} \forall t \in [0,1],\ \phi_\epsilon(t) = t, \quad \forall t \in \mathbb{R},\ -\epsilon \le \phi_\epsilon(t) \le 1+\epsilon, \\ \forall t < t',\ 0 \le \phi_\epsilon(t') - \phi_\epsilon(t) \le t' - t, \end{cases}$$

and

$$u \in \mathrm{Dom}(\mathcal{E}) \Rightarrow \phi_\epsilon(u) \in \mathrm{Dom}(\mathcal{E}),\ \mathcal{E}(\phi_\epsilon(u),\phi_\epsilon(u)) \le \mathcal{E}(u,u).$$

Remark 4. *There are two equivalent definitions of a Markovian form when $\mathcal{E}$ is closed. c.f. [27].*

Now, we say that $\mathcal{E}$ is a Dirichlet form if it is a closed Markovian form. A core of $\mathcal{E}$ is a subset of $\mathrm{Dom}(\mathcal{E}) \cap C_0(\mathcal{X})$ that is dense in $\mathrm{Dom}(\mathcal{E})$ with $\mathcal{E}_1$-norm and dense in $C_0(\mathcal{X})$ with uniform norm. Here, $C_0(\mathcal{X})$ is the space of continuous functions from $\mathcal{X}$ to $\mathbb{R}$. $\mathcal{E}$ is said to be regular if it possesses a core.
We say that $\mathcal{E}$ is local if $\forall u, v \in \mathrm{Dom}(\mathcal{E})$ such that $\mathrm{Supp}[u]$ and $\mathrm{Supp}[v]$ are disjoint compact sets, we have,

$$\mathcal{E}(u,v) = 0. \quad (5.2)$$

Lastly, it is said that the form $\mathcal{E}$ admits a carré du champ if there exists a subspace H of $\mathrm{Dom}(\mathcal{E}) \cap \mathrm{L}^\infty$, dense in $\mathrm{Dom}(\mathcal{E})$, such that

$$\forall u \in H \quad \exists \tilde{u} \in \mathrm{L}^1 \quad \forall v \in \mathrm{Dom}(\mathcal{E}) \cap \mathrm{L}^\infty(\mathcal{X},\mu), \quad 2\mathcal{E}(uv,u) - \mathcal{E}(v,u^2) = \int v\tilde{u}\,\mathrm{d}\mu.$$

The carré du champ operator then satisfies the following property:

Proposition 5.1.1. *Suppose that $\mathcal{E}$ is a local Dirichlet form admitting a carré du champ Γ. Then,*

$$\forall u \in \mathrm{Dom}[\mathcal{E}], \quad \mathcal{E}(u,u) = \frac{1}{2}\int \Gamma(u)\, d\mu.$$

5.1.2 Semigroups

Now, let $(P_t)_{t\in\mathbb{R}_+^*}$ be a semigroup of symmetric operators defined on $\mathrm{L}^2(\mathcal{X},\mu)$. That is, $(P_t)_{t\in\mathbb{R}_+^*}$ is self-adjoint, and possesses the contraction and semigroup properties. The semi-group is called strongly continuous if in addition,

$$\forall u \in \mathrm{L}^2(\mathcal{X},\mu),\ \langle P_t u - u, P_t u - u\rangle \xrightarrow[t\to 0]{} 0.$$

Definition 6 (Resolvent of the semigroup)**.** *We say that a family of linear operators* $(G_\alpha)_{\alpha\in\mathbb{R}_+^*}$ *defined on* $\mathrm{L}^2(\mathcal{X},\mu)$ *is a resolvent if*

(R1) *Each* G_α *is a self-adjoint operator defined on* $\mathrm{L}^2(\mathcal{X},\mu)$.

(R2) $G_\alpha - G_\beta + (\alpha-\beta)G_\alpha G_\beta = 0$.

(R3) $\forall\alpha>0, \forall u\in \mathrm{L}^2(\mathcal{X},\mu),\ \langle \alpha G_\alpha u, \alpha G_\alpha u\rangle \leq \langle u,u\rangle$.

The resolvent is called strongly continuous if in addition,

$$\forall u\in \mathrm{L}^2(\mathcal{X},\mu),\ \langle \alpha G_\alpha u - u, \alpha G_\alpha u - u\rangle \xrightarrow[\alpha\to\infty]{} 0.$$

Then we call resolvent of the semigroup $(P_t)_{t\in\mathbb{R}_+^*}$ *the operator defined, for* u *in* $\mathrm{L}^2(\mathcal{X},\mu)$*, by*

$$G_\alpha u = \int_0^\infty e^{-\alpha t} P_t u\, dt.$$

Remark 5. *A strongly continuous resolvent is invertible. Indeed, let* u *be such that* $G_\alpha u = 0$*. Then from* $(R2)$ *it follows that for any* β*,* $G_\beta u = 0$*. Then, since the resolvent is strongly continuous,* $u = 0$*.*

Definition 7 (Generator of a strongly continuous semigroup)**.** *The generator* $\mathcal{H}$ *of a strongly continuous semigroup is defined by*

$$\mathcal{H}u = \lim_{t\to 0}\frac{P_t u - u}{t},$$

provided this limit exists as a limit in the strong sense. The set of $u\in \mathrm{L}^2(\mathcal{X},\mu)$ *such that the limit exists is then denoted by* $\mathrm{Dom}(\mathcal{H})$*.*

Definition 8 (Generator of a resolvent)**.** *The generator* $\mathcal{H}$ *of a resolvent is defined for all* u *in* $\mathrm{L}^2(\mathcal{X},\mu)$ *by*

$$\mathcal{H}u = \alpha u - G_\alpha^{-1} u.$$

This operator is well defined since G_α is invertible and the right hand side of Au does not depend on α. A small calculation using $(R2)$ yields

$$G_{\alpha+h}^{-1} - G_\alpha^{-1} = G_\alpha^{-1}(I - hG_{\alpha+h})^{-1} h G_{\alpha+h}.$$

Then, dividing by h and letting $h\to 0$ gives the desired result.

Theorem 5.1.1 (Theorem 1.3.1 [27])**.** *There is a one to one correspondence between the family of closed symmetric forms* $\mathcal{E}$ *on* $\mathrm{L}^2(\mathcal{X},\mu)$ *and the family of nonpositive definite self-adjoint operators* $\mathcal{H}$ *on* $\mathrm{L}^2(\mathcal{X},\mu)$*. One way to put the correspondence is as follows:*

$$\begin{cases} \mathrm{Dom}(\mathcal{H}) \subset \mathrm{Dom}(\mathcal{E}), \\ \forall u\in \mathrm{Dom}(\mathcal{H}), \forall v\in \mathrm{Dom}[\mathcal{E}],\ \mathcal{E}(u,v) = \langle -\mathcal{H}u, v\rangle. \end{cases}$$

5.1.3 Dirichlet forms and Markovian Semigroups

Theorem 5.1.2 (Theorem 1.4.1 [27])**.** *Let* $\mathcal{E}$ *be a closed symmetric form on* $\mathrm{L}^2(\mathcal{X},\mu)$*. Let* (P_t) *and* (G_α) *be the strongly continuous semigroup and the strongly continuous resolvent on* $\mathrm{L}^2(\mathcal{X},\mu)$ *which are associated with* $\mathcal{E}$*. Then the next five conditions are equivalent to each other:*

(a) *P_t is Markovian for each $t > 0$.*

(b) *αG_α is Markovian for each $\alpha > 0$.*

(c) *$\mathcal{E}$ is Markovian.*

(d) *The unit contraction operates on $\mathcal{E}$.*

(e) *Every normal contraction operates on $\mathcal{E}$.*

Theorem 5.1.3 (Theorem 1.4.2 [27]). *A Dirichlet form $\mathcal{E}$ on $\mathrm{L}^2(\mathcal{X}, \mu)$ possesses the following properties:*

(i) $\forall u, v \in \mathrm{Dom}(\mathcal{E}),\ u \vee v,\ u \wedge v,\ u \wedge 1 \in \mathrm{Dom}(\mathcal{E})$.

(ii) $\forall u, v \in \mathrm{Dom}(\mathcal{E}) \cap \mathrm{L}^\infty(\mathcal{X}, \mu),\ u \cdot v \in \mathrm{Dom}(\mathcal{E})$ *and* $\sqrt{\mathcal{E}(u \cdot v, u \cdot v)} \leq \|u\|_\infty \sqrt{\mathcal{E}(v, v)} + \|v\|_\infty \sqrt{\mathcal{E}(u, u)}$.

(iii) *For u in $\mathrm{Dom}(\mathcal{E})$, define $u_n = ((-n) \vee u) \wedge n$. Then, $u_n \in \mathrm{Dom}(\mathcal{E})$ and $\|u_n - u\|_{\mathcal{E}_1} \xrightarrow[n\to\infty]{} 0$.*

Theorem 5.1.4 (Theorem 2.1.1 [27]). *Suppose $\mathcal{E}$ is a closable Markovian symmetric form on $\mathrm{L}^2(\mathcal{X}, \mu)$. Then its smallest closed extension $\bar{\mathcal{E}}$ is again Markovian and hence a Dirichlet form.*

5.1.4 Markov processes and associated semigroups

The reference for this subsection is [24]. Let $(X_t)_{t \in \mathbb{R}_+^*}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathcal{X}$. Let $\mathcal{F}_t^X = \sigma(X_s,\ s \leq t)$. Then X is a Markov process if for all $t, s > 0$,

$$\mathbb{E}[f(X_{t+s}) | \mathcal{F}_t^X] = \mathbb{E}[f(X_{t+s}) | X_t],$$

for all measurable $f : \mathcal{X} \to \mathbb{R}$. Henceforth, we consider $(X_t)_{t \in \mathbb{R}_+^*}$ a general Markov process.

Definition 9 (Time homogeneous transition function). *A function $P : [0, \infty) \times \mathcal{X} \times \mathcal{F} \to \mathbb{R}$ is a time homogeneous transition function if*

(i) $\forall (t, x) \in [0, \infty) \times X, \quad P(t, x, \cdot) \in \mathcal{P}(\mathcal{X})$.

(ii) $\forall x \in \mathcal{X}, \quad P(0, x, \cdot) = \delta_x$.

(iii) $\forall \Gamma \in \mathcal{X}, \quad P(\cdot, \cdot, \Gamma) \in \mathcal{B}([0, \infty) \times \mathcal{X})$.

(iv) $\forall s, t > 0, \forall x \in \mathcal{X}, \forall \Gamma \in \mathcal{F}, \quad P(t + s, x, \Gamma) = \int P(s, y, \Gamma) P(t, x, dy)$.

Then, a transition function P is a transition function for the time homogeneous Markov process X if

$$\forall s, t > 0, \forall \Gamma \in \mathcal{F}, \quad \mathbb{E}[f(X_{t+s}) | \mathcal{F}_t^X] = \int f(y) P(s, X_t, \mathrm{d}y),$$

for all measurable $f : \mathcal{X} \to \mathbb{R}$.
However, directly obtaining transition functions is not possible for a lot of processes. Thus, we specify a Markov process using the semigroup that it generates. In the following, $(P_t)_{t \geq 0}$ is defined by its action on an integrable $f : \mathcal{X} \to \mathbb{R}$ by

$$P_t f(x) = \int f(y) P(t, x,\ dy).$$

5.1.5 Locality, quasi-regularity

As we will see in the next subsection, it is important for a Dirichlet form to be local and quasi-regular. Hence, we define these notions rigorously here.

Definition 10 (Nest)**.** *An increasing sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of $\mathcal{N}$ (space of positive integer-valued Radon measures on E) is called an $\mathcal{E}$-nest if*

$$\bigcup_{n\in\mathbb{N}} \{F \in \mathrm{Dom}(\mathcal{E}) \mid F = 0 \ \mu\text{-a.e. on } \mathcal{N}\setminus F_n\}$$

is dense in $\mathrm{Dom}(\mathcal{E})$ *with respect to the norm induced by* $\mathcal{E}_1$*.*

A set $N \subset \mathcal{N}$ is called $\mathcal{E}$-exceptional, if there exists an $\mathcal{E}$-nest $(F_n)_{n\in\mathbb{N}}$ such that $N \subset \mathcal{N}\setminus\bigcup_{n\in\mathbb{N}} F_n$. As usual, we say that a property of points in $\mathcal{N}$ holds $\mathcal{E}$-quasi-everywhere if it holds outside of some $\mathcal{E}$-exceptional set. A function $f : \mathcal{N} \to \mathbb{R}$ is called $\mathcal{E}$-quasi-continuous if there exists an $\mathcal{E}$-nest $(F_n)_{n\in\mathbb{N}}$ such that $f_{|F_n}$ is continuous for all $n \in \mathbb{N}$.

Definition 11 (Quasi-regular Dirichlet form)**.** *A symmetric Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ on $\mathrm{L}^2(\mathcal{X}, \mu)$ is called quasi-regular if it satisfies the following:*

(QR1) *There exists an $\mathcal{E}$-nest consisting of compact sets.*

(QR2) *There exists an $\|\cdot\|_1$-dense subset of $\mathrm{Dom}(\mathcal{E})$ whose elements have $\mathcal{E}$-continuous μ-versions.*

(QR3) *There exists $(u_n)_{n\in\mathbb{N}} \in \mathrm{Dom}(\mathcal{E})^{\mathbb{N}}$ having $\mathcal{E}$-continuous μ-versions $(\tilde{u}_n)_{n\in\mathbb{N}}$, and an $\mathcal{E}$-exceptional set N such that $(\tilde{u}_n)_{n\in\mathbb{N}}$ separates points of $\mathcal{X}\setminus N$.*

We now give a lemma allowing the verification of **(QR2)** and **(QR3)** in practice.

Lemma 5.1.1. *Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a closable Dirichlet form and $(\mathcal{E}, \overline{\mathrm{Dom}(\mathcal{E})})$ its closure. Assume that D is a core of continuous functions for $(\mathcal{E}, \overline{\mathrm{Dom}(\mathcal{E})})$. Assume in addition that D separates points, i.e.*

$$\forall \xi, \eta \in \mathcal{X},\ \xi \neq \eta, \quad \exists f \in D \ / \ f(\xi) \neq f(\eta).$$

Then, **(QR2)** *and* **(QR3)** *are verified.*

Proof. By definition, D is $\|\cdot\|_1$-dense in $\overline{\mathrm{Dom}(\mathcal{E})}$. Since the functions in D are assumed to be continuous, **(QR2)** is verified. $\mathrm{L}^2(\mathcal{X}, \mu)$ is a separable Banach space, and thus a strongly Lindelöf space. Then, D is a strongly Lindelöf space (as a subspace of the previous space). Define for $\xi, \eta \in \mathcal{X}$:

$$A_{\xi,\eta} = \{f \in D \ / \ f(\xi) \neq f(\eta)\}.$$

Since D is strongly Lindelöf, then the subspace $\bigcup_{\xi,\eta\in\mathcal{X}} A_{\xi,\eta}$ is a Lindelöf space and can be written as

$$\bigcup_{\xi,\eta\in\mathcal{X}} A_{\xi,\eta} = \bigcup_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}} A_{\xi_n,\eta_m},$$

where $\xi_1, \ldots, \eta_1, \ldots \in \mathcal{X}$. Then, **(QR3)** is verified by taking $u_{n,m}$ any element in A_{ξ_n,η_m}. □

Let us also recall a way to verify **(QR1)**:

Theorem 5.1.5 (Proposition 4.1 in [49]). *Suppose there exists a bounded complete metric $\bar{\rho}$ on $\mathcal{N}$ generating the (separable) vague topology such that for all $\xi \in \mathcal{N}$, $\bar{\rho}(\cdot, \xi) \in \mathrm{Dom}(\mathcal{E})$ and $\Gamma(\bar{\rho}(\cdot, \xi)) \leq \eta$ μ-a.e. for some $\eta \in \mathrm{L}^1(\mathcal{N}, \mu)$ (independent of ξ). Then,* **(QR1)** *holds.*

And in practice, we will use the following theorem to verify quasi-regularity:

Theorem 5.1.6 (Corollary 4.9 in [49]). *Assume that condition* **(Q)**, *defined as follows, is satisfied (recall that Γ is the carré du champ operator associated with the Dirichlet form).*

(Q) *There exist $(\eta_j)_{j\in\mathbb{N}}$ in $C_0^\infty(E)$; $\eta_j \geq 0$ and $f_{l,n} : \mathcal{X} \to \mathbb{R}$ continuous such that*

- **(i)** *There exists a dense subset $(y_n)_{n\in\mathbb{N}}$ of $\mathcal{X}$ such that $\forall n \in \mathbb{N}$, $\sup_{l\in\mathbb{N}} f_{l,n} = \rho(\cdot, y_n)$.*
- **(ii)** *There exists $C \in (0, \infty)$ such that for all $j, l, n \in \mathbb{N}$ and all $\phi \in C_b^\infty(\mathbb{R})$,*

$$\eta_j \cdot (\phi \circ f_{l,n}) \in C_0^\infty(\mathcal{X}) \text{ and } \Gamma(\eta_j \cdot (\phi \circ f_{l,n})) \leq C \sup(\|\phi'\|_\infty, \|\phi\|_\infty)^2 (\eta_j + \Gamma(\eta_j)^{1/2})^2.$$

- **(iii)** *$\forall k \in \mathbb{N}$, $\exists j \in \mathbb{N}$ / $\eta_j = 1$ on $\mathcal{X}$.*

Then, $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is quasi-regular.

As a notable consequence, we also have under **(Q)** that locality holds.

Proposition 5.1.2 (Proposition 4.12 in [49]). *Assume that condition* **(Q)** *holds. Then $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ has the local property.*

Lastly, another useful set of conditions that assure locality and quasi-regularity can be found in [59]. Here $\Lambda = B(0, r)$ and set $j^i_{B(0,r)} = j^i_r$ to denote the i-th Janossy density in $B(0, r)$. We introduce the following set of hypotheses:

Hypothesis 7. **(A1)** *$(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is closable on $\mathrm{L}^2(\mathcal{X}, \mu)$*

(A2) *$\forall i \in \mathbb{N}, \forall r \geq 0$, $j^i_r \in \mathrm{L}^\infty(\mathcal{X}^i_r, dx)$ $\forall r \geq 0$, $\sum_{i=1}^\infty i\mu(\xi(B(0, r) = i)) < \infty$. Here, $\mathcal{X}^i_r := \{\xi \in \mathcal{X}_{B(0,r)} \ : \ \xi(B(0, r)) = i\}$.*

Then,

Theorem 5.1.7 (Theorem 1 in [59]). *Suppose that (A1) and (A2) hold. Then $(\mathcal{E}, \overline{\mathrm{Dom}(\mathcal{E})})$ is a local, and quasi-regular form.*

5.1.6 Diffusions associated with Dirichlet forms

To construct the diffusions associated with Dirichlet forms, we recall the results of [20], see also [48]. Indeed, we start by recalling some notions, see Chapters IV and V in [48]. Given π in the set $\mathcal{P}(\mathcal{N})$ of the probability measures on $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$, we call a π-stochastic process with state space $\mathcal{N}$,

$$\mathbf{M}_\pi = (\mathbf{\Omega}, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, (\mathbf{M}_t)_{t\geq 0}, (\mathbf{P}_\xi)_{\xi\in\mathcal{N}}, \mathbf{P}_\pi)$$

where $\mathcal{G} := \bigvee_{t\geq 0} \mathcal{G}_t$ is a σ-algebra on the set $\mathbf{\Omega}$, $(\mathcal{G}_t)_{t\geq 0}$ is the $\mathbf{P}_\pi$-completed filtration generated by the process $\mathbf{M}_t : \mathbf{\Omega} \longrightarrow \mathcal{N}$ of $(\mathcal{G}, \mathcal{B}(\mathcal{N}))$-measurable mappings, $\mathbf{P}_\xi$ is a probability measure on $(\mathbf{\Omega}, \mathcal{G})$ for all $\xi \in \mathcal{N}$, and P_π is the probability measure on $(\mathbf{\Omega}, \mathcal{G})$ defined by

$$\mathbf{P}_\pi(A) := \int_{\mathcal{N}} \mathbf{P}_\xi(A)\, \pi(\mathrm{d}\xi), \quad A \in \mathcal{G}.$$

A collection $(\mathbf{M}_\pi, (\theta_t)_{t\geq 0})$ is called a π-time homogeneous Markov process with state space $\mathcal{N}$ if $\theta_t : \mathbf{\Omega} \longrightarrow \mathbf{\Omega}$ is a shift operator, i.e. $\mathbf{M}_s \circ \theta_t = \mathbf{M}_{s+t}$, $s, t \geq 0$, the map $\xi \mapsto \mathbf{P}_\xi(A)$ is $(\mathcal{B}(\mathcal{N}), \mathcal{B}(\mathbb{R}))$-measurable for all $A \in \mathcal{G}$, and the time homogeneous Markov property

$$\mathbf{P}_\xi(\mathbf{M}_t \in A \,|\, \mathcal{G}_s) = \mathbf{P}_{\mathbf{M}_s}(\mathbf{M}_{t-s} \in A), \qquad \mathbf{P}_\xi - a.s., \quad A \in \mathcal{G}, \quad 0 \leq s \leq t, \quad \xi \in \mathcal{N}$$

holds. Recall that a π-time homogeneous Markov process $(\mathbf{M}_\pi, (\theta_t)_{t\geq 0})$ with state space $\mathcal{N}$ is said to be π-tight on $\mathcal{N}$ if $(\mathbf{M}_t)_{t\geq 0}$ is right-continuous with left limits $\mathbf{P}_\pi$-almost surely; $\mathbf{P}_\xi(\mathbf{M}_0 = \xi) = 1 \ \forall\ \xi \in \mathcal{N}$; the filtration $(\mathcal{G}_t)_{t\geq 0}$ is right continuous; the following strong Markov property holds:

$$\mathbf{P}_{\pi'}(\mathbf{M}_{t+\tau} \in A \,|\, \mathcal{G}_\tau) = \mathbf{P}_{\mathbf{M}_\tau}(\mathbf{M}_t \in A)$$

$\mathbf{P}_{\pi'}$-almost surely for all $\mathcal{G}_t$-stopping time τ, $\pi' \in \mathcal{P}(\mathcal{N})$, $A \in \mathcal{G}$ and $t \geq 0$, cf. Theorem IV.1.15 in [48]. In addition, a π-tight process on $\mathcal{N}$ is said π-special standard process on $\mathcal{N}$ if for any $\pi' \in \mathcal{P}(\mathcal{N})$ which is equivalent to π and all $\mathcal{G}_t$-stopping times τ, $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \tau$ then $\mathbf{M}_{\tau_n}$ converges to $\mathbf{M}_\tau$, $\mathbf{P}_{\pi'}$-almost surely.

The following theorem holds, in which $\mathbb{E}_\xi$ denotes the expectation under $\mathbf{P}_\xi$, $\xi \in \mathcal{N}$.

Theorem 5.1.8. *Assume that the Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is closable, and that its closure $(\overline{\mathcal{E}}, \overline{\mathrm{Dom}(\mathcal{E})})$ is quasi-local and regular. In the following, we call $\mathcal{H}$ the generator associated to the Dirichlet form. Then there exists a μ-tight special standard process $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$ on $\mathcal{N}$ with transition semigroup*

$$p_t f(\xi) := \mathbb{E}_\xi[f(\mathbf{M}_t)], \quad \xi \in \mathcal{N}, \quad f : \mathcal{N} \longrightarrow \mathbb{R} \quad \textit{square integrable.}$$

In addition, $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$ is properly associated with the Dirichlet form $(\overline{\mathcal{E}}, \overline{\mathrm{Dom}(\mathcal{E})})$ in the sense that $p_t f$ is an $\overline{\mathcal{E}}$-a.c., μ-version of $\exp(t\mathcal{H})f$ for all square integrable $f : \mathcal{N} \longrightarrow \mathbb{R}$ and $t > 0$, and such that

$$\mathbf{P}_\xi(\{\omega :\ t \mapsto \mathbf{M}_t(\omega) \textit{ is continuous on } [0, +\infty)\}) = 1, \qquad \overline{\mathcal{E}}\textit{-a.e., } \xi \in \mathcal{N}, \tag{5.3}$$

i.e. $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$ is quasi-continuous.

Proof. The Dirichlet form $(\overline{\mathcal{E}}, \overline{\mathrm{Dom}(\mathcal{E})})$ is quasi-regular, hence by Theorem III.3.5 page 103 in [48], there exists a μ-tight special standard process on $\mathcal{N}$, say $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$, whose transition semigroup $(p_t)_{t\geq 0}$ is such that, for any square integrable function $f : \mathcal{N} \longrightarrow \mathbb{R}$ and $t > 0$, $p_t f$ is a μ-version of $\exp(t\mathcal{H})f$ and $p_t f$ is $\overline{\mathcal{E}}$-a.c.. Since the form has the local property, by Theorem V.1.5 page 150 in [48] the tight special standard process is a μ-diffusion associated to the form, i.e. relation (B.5.1) holds for $\overline{\mathcal{E}}$-a.e. $\xi \in \mathcal{N}$. □

For the sake of completeness we remark that by applying Theorem 6.4 page 141 in [49], one has that the μ-diffusion on $\mathcal{N}$ properly associated with the Dirichlet form $(\overline{\mathcal{E}}, \overline{\mathrm{Dom}(\mathcal{E})})$ (defined in Theorem B.5.1) is unique up to μ-equivalence. We refer the reader to Definition 6.3 page 140 in [49] for the meaning of this notion.
Moreover, the Dirichlet form is associated with a Markov process with nice sample paths (i.e. quasi-continuous) if and only if this form is quasi-regular. This means that quasi-regularity is the correct notion in this context.

Another characterization of the diffusion is the following: ($\tilde{G}$ denotes a quasi-continuous version of G, which exists since the form is quasi-regular.)

Theorem 5.1.9 (Diffusion process as a solution of a martingale problem)**.** $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$ *from the previous theorem is the (up to* $\mu_{-1,K}$*-equivalence) unique diffusion process having* μ *as an invariant measure and solving the following martingale problem:* $\forall G \in \mathrm{Dom}(\mathcal{H})$*,*

$$\forall t \geq 0 \ \tilde{G}(\mathbf{X_t}) - \tilde{G}(\mathbf{X_0}) + \int_0^t \mathcal{H}\, G(\mathbf{X_s})\, ds$$

is an $\mathcal{G}$*-martingale under* $\mathbf{P}_\xi$ *for* $\mathcal{E}$*-q.e.* $\xi \in \mathcal{X}$*. Here,* $\mathcal{H}$ *is the generator of the semigroup* p_t *(defined in Definition 7).*

To summarize, once we prove that a Dirichlet form is local, and quasi-regular, we know that there exists an associated Markov process which has interesting properties. Indeed, it is right-continuous, and satisfies the martingale problem (which leads to a characterization in terms of a Stochastic Differential Equation).

5.1.7 Capacity

Let us introduce a notion of capacity of the Dirichlet space $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$. Recall that we have denoted by $\mathcal{O}(\mathcal{X})$ the set consisting of all non-empty open sets in $\mathcal{X}$. For $O \in \mathcal{O}(\mathcal{X})$, define

$$\mathrm{L}_O = \{F \in \mathrm{Dom}(\mathcal{E}) \ / \ F \geq 1 \quad \mu \text{ - a.e. on } O\}.$$

Then we define the capacity of O as (c.f. [27])

$$\mathrm{Cap}(O) = \begin{cases} \inf_{F \in \mathrm{L}_O} \mathcal{E}_1(F,F) & \mathrm{L}_O \neq \emptyset, \\ \infty & \mathrm{L}_O = \emptyset, \end{cases}$$

where we have used the notation $\mathcal{E}_1(F,F) := \mathcal{E}(F,F) + \langle F, F\rangle_{\mathrm{L}^2(\mathcal{X},\mu)}$. Then, for an arbitrary set $X \subset \mathcal{X}$, we also set $\mathrm{Cap}(X) = \inf_{X \subset O \in \mathcal{O}(\mathcal{X})} \mathrm{Cap}(O)$. It should be noted that the notion of capacity is dependent on the underlying Dirichlet form. Hence, we shall call Cap the 1-capacity of the (pre-) Dirichlet space $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$. We also sometimes write $\mathrm{Cap}_{\mathcal{E}}(X)$ to insist on the fact that the capacity is the one induced by $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$.

We now recall a few basic properties of the previously defined capacity, which can be found in [27].

Proposition 5.1.3. *For each* $A \in \mathcal{O}(\mathcal{X})$*, there exists a unique element* $e_A \in \mathrm{L}_A$ *such that*

(i) $\mathcal{E}_1(e_A, e_A) = \mathrm{Cap}(A)$.

(ii) $0 \leq e_A \leq 1$ μ*-a.e. and* $e_A = 1$ μ*-a.e. on* A.

(iii) e_A *is the unique element of* $\mathrm{Dom}(\mathcal{E})$ *satisfying* $e_A = 1$ P*-a.e. on* A *and* $\forall v \in \mathrm{Dom}(\mathcal{E})$ *such that* $v \geq 0$ μ*-a.e. on* A*,* $\mathcal{E}_1(e_A, v) \geq 0$.

(iv) $\forall v \in \mathrm{Dom}(\mathcal{E})$*,* $v = 1$ μ*-a.e. on* $A \Rightarrow \mathcal{E}_1(e_A, v) = \mathrm{Cap}(A)$.

(v) $\forall A, B \in \mathcal{O}(\mathcal{X}), \quad A \subset B \Rightarrow e_A \leq e_B$ μ*-a.e.*

Then, the capacity defined as above is a Choquet capacity, in the sense that:

Proposition 5.1.4. (i) $\forall A, B \in \mathcal{O}(\mathcal{X}), \quad A \subset B \Rightarrow \mathrm{Cap}(A) \leq \mathrm{Cap}(B)$.

(ii) $\forall (A_n)_{n\in\mathbb{N}} \in \mathcal{O}(\mathcal{X})^{\mathbb{N}}$ *such that* (A_n) *is increasing,* $\mathrm{Cap}(\bigcup_n A_n) = \sup_n \mathrm{Cap}(A_n)$.

(iii) $\forall (A_n)_{n\in\mathbb{N}} \in \mathcal{O}(\mathcal{X})^{\mathbb{N}}$ *such that* (A_n) *is a decreasing sequence of compacts,* $\mathrm{Cap}(\bigcap_n A_n) = \inf_n \mathrm{Cap}(A_n)$.

An important result that links exceptional sets and capacity is given by Theorem 4.3.1 in [27].

Proposition 5.1.5. *A set $N \subset \mathcal{X}$ is ($\mathcal{E}-$)exceptional if and only if* $\mathrm{Cap}(N) = 0$.

In particular, the Markov process associated with a Dirichlet form hits a set $N \subset \mathcal{X}$ if and only if $\mathrm{Cap}(N) \neq 0$.

5.1.8 Convergence of Dirichlet forms

In this subsection, we study which properties transfer from a smaller Dirichlet form to a larger one, and vice versa. Let us begin with a characterization of $\mathcal{E}$-nests.

Proposition 5.1.6 (Theorem 2.14 in [47])**.** *An increasing sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets is an $\mathcal{E}$-nest if and only if*

$$\mathrm{Cap}_{\mathcal{E}}(F_n^c) \xrightarrow[n\to\infty]{} 0,$$

where A^c is the complement set of $A \in \mathcal{X}$.

We move on to prove that quasi-regularity transfers itself to the larger form.

Proposition 5.1.7. *Let $(\mathcal{E}_1, D_1)$ and $(\mathcal{E}_2, D_2)$ be two Dirichlet forms such that $(\mathcal{E}_2, D_2)$ extends $(\mathcal{E}_1, D_1)$. Assume that $(\mathcal{E}_1, D_1)$ is quasi-regular. Then $(\mathcal{E}_2, D_2)$ is also quasi-regular.*

Proof. It is immediate that for any $X \in \mathcal{X}$, $\mathrm{Cap}_{\mathcal{E}_1}(X) \geq \mathrm{Cap}_{\mathcal{E}_2}(X)$. Then, Proposition 5.1.5 implies that $(\mathcal{E}_2, D_2)$ verifies **(QR1)**. The two other properties follow directly from the definition. □

Remark 6. *In particular, this means that the study of the properties of a Dirichlet form with a smaller domain generalize to bigger domains. Indeed, there are more exceptional sets in the first setting, meaning that the proofs are harder.*

Next, we give two convergence results that ensure the conservation of the closability of the Dirichlet forms. For two Dirichlet forms $(\mathcal{E}_1, D_1)$ and $(\mathcal{E}_2, D_2)$, we say that $\mathcal{E}_1 \leq \mathcal{E}_2$ if

$$D_2 \subset D_1 \text{ and } \forall F \in D_2,\ \mathcal{E}_1(F) \leq \mathcal{E}_2(F). \tag{5.4}$$

We say that a sequence $(\mathcal{E}^n, \mathcal{D}^n)_{n\in\mathbb{N}}$ of Dirichlet forms is increasing if $\mathcal{E}_n \geq \mathcal{E}_m$ when $n > m$.

Lemma 5.1.2 (Prop I.3.7, (i) in [48])**.** *Suppose $(\mathcal{E}^n, \mathcal{D}^n)_{n\in\mathbb{N}}$ is an increasing sequence of positive definite symmetric bilinear forms on $\mathrm{L}^2(\mathcal{X}, \mu)$. Moreover, assume $(\mathcal{E}^n, \mathcal{D}^n)_{n\in\mathbb{N}}$ are closable. Define*

$$\begin{cases} \mathcal{E}^\infty(f,f) = \lim_{n\to\infty} \mathcal{E}^n(f,f), \\ \mathrm{Dom}(\mathcal{E}^\infty) = \mathcal{D}^\infty = \{f \in \cap_{n\in\mathbb{N}} \mathcal{D}^n : \sup \mathcal{E}^n(f,f) < \infty\}. \end{cases}$$

Then, $(\mathcal{E}^\infty, \mathcal{D}^\infty)$ is closable.

Lemma 5.1.3 (Prop I.3.7 [48])**.** *Suppose $(\mathcal{E}^n, \mathcal{D}^n)_{n\in\mathbb{N}}$ is an increasing sequence of positive definite symmetric bilinear forms on $\mathrm{L}^2(\mathcal{X}, \mu)$. Moreover, assume $(\mathcal{E}^n, \mathcal{D}^n)_{n\in\mathbb{N}}$ are closed. Define*

$$\begin{cases} \mathcal{E}^\infty(f,f) = \lim_{n\to\infty} \mathcal{E}^n(f,f), \\ \mathrm{Dom}(\mathcal{E}^\infty) = \mathcal{D}^\infty = \{f \in \cap_{n\in\mathbb{N}} \mathcal{D}^n : \sup \mathcal{E}^n(f,f) < \infty\}. \end{cases}$$

Then, $(\mathcal{E}^\infty, \mathcal{D}^\infty)$ is closed and the resolvent G_α^n converge to G_α^∞ strongly in $\mathrm{L}^2(\mathcal{X}, \mu)$ for all $\alpha > 0$.

5.2 Differential gradient

We use here some of the notations of [88], even though these notations are quite widespread. Let $\mathrm{V}(E)$ be the set of C^∞ vector fields on E. Let $\mathrm{V}_0(E)$ be the set of C^∞ vector fields on E with compact support. Define $(\phi_t^v)_{T\geq 0}$ as the solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\phi_t^v(x) = v(\phi_t^v(x)), \\ \phi_0^v(x) = x. \end{cases}$$

For any $\xi \in \mathcal{X}$, we still denote by ϕ_t^v the map from $\mathcal{X}$ into $\mathcal{X}$. For a function $F : \mathcal{X} \longrightarrow \mathbb{R}$, the directional derivative at $\xi \in \mathcal{X}$ along the vector field $v \in V_0(E)$ is defined by

$$\nabla_v F(\xi) = \frac{\mathrm{d}}{\mathrm{d}t} F(\phi_t^v(\xi))\Big|_{t=0}.$$

A function $F : \mathcal{X} \to \mathbb{R}$ is said to be differentiable at $\xi \in \mathcal{X}$ if for any vector field $v \in V_0(E)$, the directional derivative along the vector field v is well defined.

The intrinsic gradient of F is the mapping

$$\xi \longmapsto (\nabla . F)(\xi),$$

such that, for any $v \in V_0(E)$,

$$\nabla_v F(\xi) = \int \langle \nabla_x F(\xi), v(x)\rangle_E \, \xi(\mathrm{d}x).$$

In practice, we know how to calculate the directional derivative in the case of cylindrical functions. We use the notations from [3]. That is, we say that a function $F : \mathcal{X} \to \mathbb{R}$ is in $\mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})$ if F is of the form

$$F : \xi \longmapsto f(\int h_1 \, \mathrm{d}\xi, \ldots, \int h_N \, \mathrm{d}\xi),$$

for some $N \in \mathbb{N}$, $h_1, \ldots, h_N \in \mathcal{D} = C^\infty(E)$, $f \in C_b^\infty(\mathbb{R}^N)$.

Proposition 5.2.1. *The class $\mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})$ is dense in $\mathrm{L}^2(\mathcal{X}, \mu)$.*

Proof. The proof essentially uses the Hahn-Banach theorem, see [75]. □

We can then evaluate the directional derivative for a function F that is of such form. For any $v \in V_0(E)$,

$$\begin{aligned} F(\phi_t^v(\xi)) &= f(\sum_{x\in\phi_t^v(\xi)} h_1(x), \ldots, \sum_{x\in\phi_t^v(\xi)} h_N(x)) \\ &= f(\sum_{x\in\xi} h_1(\phi_t^v(x)), \ldots, \sum_{x\in\xi} h_N(\phi_t^v(x)) \\ &= f(\int h_1 \circ \phi_t^v \, d\xi, \ldots, \int h_N \circ \phi_t^v \, d\xi). \end{aligned}$$

In that case, we have a simple expression of the directional gradient:

$$\nabla_v F(\xi) = \sum_{i=1}^N \partial_i f(\int h_1 \, \mathrm{d}\xi, \ldots, \int h_N \, \mathrm{d}\xi) \int \nabla_v^E h_i \, \mathrm{d}\xi.$$

Note that we can write

$$\int \nabla_v^E h_i \, \mathrm{d}\xi = \int \langle \nabla^E h_i, v \rangle_E \, \mathrm{d}\xi,$$

so the intrinsic gradient of F at $\xi \in \mathcal{X}$ is

$$\nabla_x F(\xi) = \sum_{i=1}^{N} \partial_i f(\int h_1 \, \mathrm{d}\xi, \dots, \int h_N \, \mathrm{d}\xi) \nabla^E h_i(x).$$

We also introduce the set of test functions used in [20].

Definition 12. *A function $F : \mathcal{X} \to \mathbb{R}$ is in $\mathcal{S}$ if*

$$F(\xi) = f_0 \mathbf{1}_{\{\xi(E)=0\}} + \sum_{k=1}^{n} \mathbf{1}_{\{\xi(E)=k\}} f_k(X_1, \dots, X_k), \tag{5.5}$$

where $n \geq 1$ is an integer, for any $k = 1, \dots, n$, $f_k \in \mathcal{C}^\infty(E^k)$ is a symmetric function and $f_0 \in \mathbb{R}$ is a constant. Similarly, when $F : \mathcal{X}_\Lambda \to \mathbb{R}$ is of the form (5.5), then we say that F is in $\mathcal{S}_\Lambda$.

Remark 7. *Conditionally on the number of points in ξ being less than n, a function in $\mathcal{S}$ is also in $\mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X})$.*

However, both of the functional sets are dense in $\mathrm{L}^2(\mathcal{X}, \mu)$, as the following lemma shows:

Lemma 5.2.1. *$\mathcal{S}$ is dense in $\mathrm{L}^2(\mathcal{X}, \mu)$.*

Proof. See [20]. □

Lastly, we define the Dirichlet form associated with a point process μ on $(\mathcal{X}, \mathcal{F})$.

Definition 13 (Pre-Dirichlet form)**.** *For $F, G \in \mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X})$, we introduce the following positive definite symmetric form*

$$\mathcal{E}(F, G) = \frac{1}{2} \int_{\mathcal{X}} \int \langle \nabla_x F(\xi), \nabla_x G(\xi) \rangle \, \xi(dx) \, \mu(d\xi), \tag{5.6}$$

with domain $\mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X}) \subset \mathrm{L}^2(\mathcal{X}, \mu)$.

Proposition 5.2.2. *Assume that μ has finite moments, i.e. for all φ bounded, compactly supported function, and for all $n \in \mathbb{N}$,*

$$\mathbb{E}[(\int \varphi(y) \, \xi(dy))^n] < \infty. \tag{5.7}$$

Then, the form $(\mathcal{E}, \mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X}))$ is well-defined.

Proof. For $F \in \mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X})$, for $\xi \in \mathcal{X}$,

$$\int_{\mathcal{X}} \int \langle \nabla_x F, \nabla_x G \rangle \, \xi(\mathrm{d}x) = \sum_{i,j=1}^{n} \partial_{i,j} f(\int h_1 \mathrm{d}\xi, \dots, \int h_N \mathrm{d}\xi) \int_E \langle \nabla^E h_i(x), \nabla^E h_j(x) \rangle_E \, \xi(\mathrm{d}x),$$

so if we note $\phi_{i,j} : x \mapsto \langle \nabla^E h_i(x), \nabla^E h_j(x) \rangle_E$,

$$|\int_{\mathcal{X}} \int \langle \nabla_x F, \nabla_x G \rangle \, \xi(\mathrm{d}x)| \leq C \sum_{i,j=1}^{n} \int_{\mathcal{X}} |\langle \phi_{i,j}, \xi \rangle| \, \mu(\mathrm{d}\xi).$$

Since $\phi_{i,j}$ is compactly supported, the right hand side of the previous inequality is finite when μ verifies (5.7). □

Lastly, we also define the Dirichlet form associated to the measure μ_Λ, i.e. μ restricted to a compact subset $\Lambda \subseteq E$.

Definition 14 (Pre-Dirichlet form on a compact set)**.** *For $F, G \in \mathcal{FC}_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda)$, we introduce the following positive definite symmetric form*

$$\mathcal{E}_\Lambda(F,G) = \frac{1}{2}\int_{\mathcal{X}_\Lambda}\int \langle \nabla_x F, \nabla_x G\rangle\, \xi(dx)\, \mu_\Lambda(d\xi),$$

with domain $\mathcal{FC}_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda) \subset \mathrm{L}^2(\mathcal{X}, \mu_\Lambda)$. The Dirichlet form thus defined is well-defined in the sense that $\mathcal{E}_\Lambda(\mathcal{FC}_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda), \mathcal{FC}_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda)) \subset \mathbb{R}$, by the same arguments as in Proposition 5.2.2 and no additional hypothesis on μ .

5.3 Determinantal point process

In this section[1], we study the case of the determinantal point process, i.e. the α-determinantal point process with $\alpha = -1$. Throughout this section, we assume that Hypothesis 3 is verified.

5.3.1 Quasi-Invariance

In this section, we recall some results from [14] and make some of the proofs more precise. Let $\mathrm{Diff}_0(E)$ be the set of all diffeomorphisms from E into itself with compact support, i.e., for any $\phi \in \mathrm{Diff}_0(E)$, there exists a compact outside of which ϕ is the identity map. In particular, note that for a compact subset $\Lambda \subseteq E$, $\mathrm{Diff}_0(\Lambda)$ is the set of diffeomorphisms from Λ into itself. For any reference measure λ on E, λ_ϕ denotes the image measure of λ by ϕ. For $\phi \in \mathrm{Diff}_0(E)$ whose support is included in Λ, we introduce the isometry $\mathcal{T}_\phi$,

$$\begin{aligned}\mathcal{T}_\phi \,:\, \mathrm{L}^2(\Lambda, \lambda_\phi) &\longrightarrow \mathrm{L}^2(\Lambda, \lambda)\\ f &\longmapsto f\circ\phi.\end{aligned}$$

Its inverse, which exists since ϕ is a diffeomorphism, is trivially defined by $f \mapsto f\circ\phi^{-1}$ and denoted by $\mathcal{T}_\phi^{-1}$. Note that $\mathcal{T}_\phi$ and $\mathcal{T}_\phi^{-1}$ are isometries, i.e.,

$$<\mathcal{T}_\phi\psi_1,\ \mathcal{T}_\phi\psi_2>_{\mathrm{L}^2(\Lambda,\lambda)} = <\psi_1,\ \psi_2>_{\mathrm{L}^2(\Lambda,\lambda_\phi)},$$

for any ψ_1 and ψ_2 belonging to $\mathrm{L}^2(\Lambda, \lambda_\phi)$. We also set:

$$\mathcal{K}_\Lambda^\phi = \mathcal{T}_\phi^{-1}\mathcal{K}_\Lambda\mathcal{T}_\phi \text{ and } \mathcal{J}^\phi[\Lambda] = \mathcal{T}_\phi^{-1}\mathcal{J}[\Lambda]\mathcal{T}_\phi.$$

Lastly, for any $\alpha = \{x_n\}_{1\le n\le\alpha(E)} \in \mathcal{X}$, we denote by Φ the map:

$$\begin{aligned}\Phi \,:\, \mathcal{X} &\longrightarrow \mathcal{X}\\ \{x_n\}_{1\le n\le\alpha(E)} &\longmapsto \{\phi(x_n)\}_{1\le n\le\alpha(E)}.\end{aligned}$$

1. I present here my approach to the proof of the existence of diffusions associated with determinantal point processes. The steps detailed in this section were obtained during the first year of my PhD study. During that time, it came to our attention that N. Privault and G. L. Torrisi had independently obtained similar results. After fruitful discussions, we agreed to cosign the paper [20]. In this section, I emphasize my own contributions to the paper, and present the results in a personal way. The approach of this section is therefore sometimes different than that of [20] which is why I have also chosen to put in Appendix B the theorems and proofs as they were given in [20].

With the previous definitions in mind, we can give the following result from [14]:

Lemma 5.3.1. *Let $\mathcal{K}$ be an operator satisfying Hypothesis 3, and $\mathcal{J}[\Lambda]$ the associated local interaction operator. We have the following properties.*

a) $\mathcal{K}_\Lambda^\phi$ *and* $\mathcal{J}^\phi[\Lambda]$ *are continuous operators from* $\mathrm{L}^2(\Lambda, \lambda_\phi)$ *into* $\mathrm{L}^2(\Lambda, \lambda_\phi)$.

b) $\mathcal{K}_\Lambda^\phi$ *is of trace class and* $\mathrm{Tr}(\mathcal{K}_\Lambda^\phi) = \mathrm{Tr}(\mathcal{K}_\Lambda)$.

c) $\mathrm{Det}(\mathrm{I} - \mathcal{K}_\Lambda^\phi) = \mathrm{Det}(\mathrm{I} - \mathcal{K}_\Lambda)$. *This translates into the fact that* $P(\xi(\Lambda) = 0) = P(\Phi(\xi)(\Lambda) = 0)$ *which is expected since ϕ is a diffeomorphism.*

d) $\mathcal{K}_\Lambda^\phi$ *is again an integral operator on* $\mathrm{L}^2(\Lambda, \lambda_\phi)$ *whose kernel is given by*

$$(x,\, y) \mapsto K_\Lambda\left(\phi^{-1}(x), \phi^{-1}(y)\right).$$

An analog formula also holds for the operator $\mathcal{J}^\phi[\Lambda]$, i.e. its kernel $J^\phi[\Lambda]$ is given by $(x,\, y) \mapsto J[\Lambda]\left(\phi^{-1}(x), \phi^{-1}(y)\right)$.

e) $\mathcal{J}^\phi[\Lambda]$ *is correctly defined as the local interaction operator associated with* $\mathcal{K}_\Lambda^\phi$, *i.e.* $(\mathrm{I} - \mathcal{K}_\Lambda^\phi)^{-1}\mathcal{K}_\Lambda^\phi = \mathcal{J}^\phi[\Lambda]$.

Proof. The proof can be found in [14], except for point *e*), which we now prove. We have by definition

$$\begin{aligned}(\mathrm{I} - \mathcal{K}_\Lambda^\phi)^{-1}\mathcal{K}_\Lambda^\phi &= (\mathrm{I} - \mathcal{T}_\phi^{-1}\mathcal{K}_\Lambda\mathcal{T}_\phi)^{-1}\mathcal{T}_\phi^{-1}\mathcal{K}_\Lambda^\phi\mathcal{T}_\phi \\ &= \mathcal{T}_\phi^{-1}(\mathrm{I} - \mathcal{K}_\Lambda)^{-1}\mathcal{K}_\Lambda\mathcal{T}_\phi \\ &= \mathcal{J}^\phi[\Lambda],\end{aligned}$$

which is the aforementioned result. □

Then, it is known how the map ϕ transforms a determinantal process. More precisely,

Lemma 5.3.2 (Theorem 7 in [14])**.** *Let $\mathcal{K}_\Lambda$ be an operator satisfying Hypothesis 3, and μ be the associated determinantal process. Then, $\Phi^*\mu$ is again determinantal with kernel $\mathcal{K}_\Lambda^\phi$ with respect to the measure λ_ϕ on E.*

To prove the quasi-invariance of the determinantal measure restricted to a compact subset $\Lambda \subseteq E$ with respect to the group of diffeomorphisms on E, we still need one last result:

Lemma 5.3.3. *Let $\mathcal{K}$ be an operator satisfying Hypothesis 3, and $\mathcal{J}[\Lambda]$ the associated local interaction operator, with kernel $J[\Lambda]$. Assume that $\|\mathcal{K}_\Lambda\| < 1$. Then, we have that* $\det J[\Lambda](\xi) > 0$, *for μ_Λ-a.e. $\xi \in \mathcal{X}$. However, we do not in general have* $\det J[\Lambda](\alpha) > 0$, *for L_Λ^λ-a.e. $\alpha \in \mathcal{X}$, where L_Λ^λ is the sample measure defined in (3.1).*

Proof. Recall that for a determinantal process with kernel $\mathcal{K}$ satisfying Hypothesis 3, we have $\mu_\Lambda \ll L_\Lambda^\lambda$ and $j_\Lambda = \dfrac{\mathrm{d}\mu_\Lambda}{\mathrm{d}L_\Lambda^\lambda}$. Moreover, it is known in the determinantal case that (3.9) holds, i.e.,

$$j_\Lambda(\alpha) = \mathrm{Det}(\mathrm{I} - \mathcal{K}_\Lambda)\det J[\Lambda](\alpha),$$

for $\alpha \in \mathcal{X}_\Lambda$. Since j_Λ is a density, we obviously have that $j_\Lambda(\alpha) > 0$, for μ_Λ-a.e. $\alpha \in \mathcal{X}_\Lambda$. Hence, since $\|\mathcal{K}_\Lambda\| < 1$, we have $\det J[\Lambda](\alpha) > 0$, for μ_Λ-a.e. $\alpha \in \mathcal{X}_\Lambda$. As to the concluding

part of the lemma, we notice that in general, one does not have $\mu_\Lambda \sim L^\lambda_\Lambda$. Indeed, consider for example the case where $\mathrm{Rank}(\mathcal{K}_*) \leq N \in \mathbb{N}^*$. Then, $j^{N+1}_\Lambda(x_1, \dots, x_{N+1}) = 0$, for $\mu^{\otimes(N+1)}$-a.e. $(x_1, \dots, x_{N+1}) \in \Lambda^{N+1}$ (since μ_Λ has less than N points almost surely, see Proposition 3.3.3). It suffices to define the set

$$A := \{B \subseteq \Lambda : \ |B| = N + 1\},$$

which verifies $\mu_\Lambda(A) = 0$ but $L^\lambda_\Lambda(A) = \frac{1}{n!}\lambda(\Lambda)$. □

Remark 8. *If we suppose that, for any $n \geq 1$, the function*

$$(x_1, \dots, x_n) \longmapsto \det J[\Lambda](x_1, \dots, x_n)$$

is strictly positive $\lambda^{\otimes n}$-a.e. on Λ^n, then we have $\mu_\Lambda \sim L^\lambda_\Lambda$ and therefore $\det J[\Lambda](\alpha) > 0$, for L^λ_Λ-a.a. $\alpha \in \mathcal{X}_\Lambda$.

We are now ready to give the main result of this section. We emphasize that despite the result being the same as the one in [14], the proof given there implicitly uses the fact that $\det J[\Lambda](\alpha) > 0$, for L^λ_Λ-a.a. $\alpha \in \mathcal{X}_\Lambda$, which is known not to be true in general. Henceforth, we shall also assume the following technical condition which ensures that there exists an integration by parts formula on the underlying space E.

Hypothesis 8. *The Radon measure λ is absolutely continuous* w.r.t. *the Lebesgue measure ℓ on E, with Radon-Nikodym derivative $\rho = \frac{\mathrm{d}\lambda}{\mathrm{d}\ell}$ which is strictly positive and continuously differentiable on E.*

Then for any $\phi \in \mathrm{Diff}_0(E)$, λ_ϕ is absolutely continuous with respect to λ with density given by

$$p^\lambda_\phi(x) = \frac{\mathrm{d}\lambda_\phi(x)}{\mathrm{d}\lambda(x)} = \frac{\rho(\phi^{-1}(x))}{\rho(x)}\mathrm{Jac}(\phi^{-1})(x), \tag{5.8}$$

where $\mathrm{Jac}(\phi^{-1})(x)$ is the Jacobian of ϕ^{-1} at point $x \in E$. We draw attention to the fact that it is indeed $\mathrm{Jac}(\phi^{-1})(x)$ that appears in (5.8), which contradicts equation (2.11) of [2]. We can now state the main result of this section:

Proposition 5.3.1. *Let $\mathcal{K}_\Lambda$ be a restriction operator satisfying Hypothesis 3. Assume that Hypothesis 8 is verified. Then, for any measurable nonnegative f on Λ:*

$$\mathbb{E}\left[e^{-\sum_{x\in\xi_\Lambda} f\circ\phi(x)}\right] = \mathbb{E}\left[e^{-\sum_{x\in\xi_\Lambda} f(x)} e^{\sum_{x\in\xi_\Lambda} \ln(p^\lambda_\phi(x))} \ \frac{\det J^\phi[\Lambda](\xi_\Lambda)}{\det J[\Lambda](\xi_\Lambda)}\right]. \tag{5.9}$$

Remark 9. *The right-hand side of (5.9) is well defined thanks to lemma 5.3.3.*

Remark 10. *Interpretations and additional remarks can be found in [14, Theorem 8].*

Proof. For any measurable nonnegative f on Λ, we have since $\det J[\Lambda] > 0$ μ_Λ-a.e.:

$$\begin{aligned}
&\mathbb{E}\left[e^{-\sum_{x\in\xi_\Lambda} f(x)} e^{\sum_{x\in\xi_\Lambda} \ln(p_\phi^\lambda(x))} \frac{\det J^\phi[\Lambda](\xi_\Lambda)}{\det J[\Lambda](\xi_\Lambda)}\right] \\
&= \mathbb{E}\left[e^{-\sum_{x\in\xi_\Lambda} f(x)} e^{\sum_{x\in\xi_\Lambda} \ln(p_\phi^\lambda(x))} \frac{\det J^\phi[\Lambda](\xi_\Lambda)}{\det J[\Lambda](\xi_\Lambda)} \mathbf{1}_{\{\det J[\Lambda]>0\}}\right] \\
&= \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} e^{-\sum_{k=1}^n f(x_k)} \prod_{k=1}^n p_\phi^\lambda(x_k) \\
&\qquad\qquad \frac{\det J^\phi[\Lambda](x_1,\dots,x_n)}{\det J[\Lambda](x_1,\dots,x_n)} j_\Lambda(x_1,\cdots,x_n) \mathbf{1}_{\{j_\Lambda(x_1,\cdots,x_n)>0\}} \lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_n) \\
&= \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} e^{-\sum_{k=1}^n f(x_k)} \prod_{k=1}^n p_\phi^\lambda(x_k) \det J^\phi[\Lambda](x_1,\dots,x_n) \operatorname{Det}(\mathrm{I}-\mathcal{K}_\Lambda) \lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_n) \\
&= \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} e^{-\sum_{k=1}^n f(x_k)} \operatorname{Det}(\mathrm{I}-\mathcal{K}_\Lambda^\phi) \det J^\phi[\Lambda](x_1,\dots,x_n) \lambda_\phi(\mathrm{d}x_1)\dots\lambda_\phi(\mathrm{d}x_n),
\end{aligned}$$

where we have used (3.9), (5.8) and Lemma 5.3.1, c). Then, we conclude by Lemma 5.3.2. Indeed, $\operatorname{Det}(\mathrm{I}-\mathcal{K}_\Lambda^\phi) \det J^\phi[\Lambda](x_1,\dots,x_n)$ is the Janossy density of $\Phi^*\mu$ with respect to $L_\Lambda^{\lambda_\phi}$(see lemma 5.3.1, e)), which yields (5.9). □

5.3.2 Integration by parts

In this subsection, we recall the main result from [14] which was proved by taking a different approach in [20]. Here, we restrict our attention to a compact set $\Lambda \subseteq E$, and obtain an integration by parts formula for the point process μ_Λ on $(\mathcal{X}_\Lambda, \mathcal{F}_\Lambda)$.
We start by defining $U[\Lambda] := \xi \mapsto -\ln\det J[\Lambda](\xi)$. Note that by Lemma 5.3.3, U is well-defined for μ_Λ-almost every $\xi \in \mathcal{X}_\Lambda$. We consider the additional hypothesis on $J[\Lambda]$:

Hypothesis 9. *Assume that for all compact sets $\Lambda \subseteq E$, the kernel $J[\Lambda](\cdot,\cdot)$ is continuously differentiable on $\Lambda\times\Lambda$.*

The previous condition implies in particular that $U[\Lambda]$ is itself continuously differentiable, by differentiability of the determinant.
There are considerable changes depending on the measure we choose on E. More precisely, let as previously ℓ be the Lebesgue measure on E, and assume that λ is absolutely continuous with respect to ℓ. Note

$$\rho = \frac{\mathrm{d}\lambda}{\mathrm{d}\ell},$$

and assume that ρ satisfies Hypothesis 8. Then, let us note β^λ the logarithmic derivative of λ, given by:

$$\forall x \in E,\ \beta^\lambda(x) = \begin{cases} \frac{\nabla\rho(x)}{\rho(x)} \text{ on } \{\rho(x)>0\}, \\ 0 \text{ on } \{\rho(x)=0\}. \end{cases}$$

Then, for any ϕ_1, ϕ_2 in $\mathcal{D}$ (the domain on which the gradient ∇^E is defined on E), we have the integration by parts formula on E

$$\int_E \nabla_v^E \phi_1(x)\phi_2(x)\,\lambda(\mathrm{d}x) = -\int_E \phi_1(x)\nabla_v^E\phi_2(x)\,\lambda(\mathrm{d}x) - \int_E \phi_1(x)\phi_2(x)\beta_v^\lambda(x)\,\lambda(\mathrm{d}x), \quad (5.10)$$

where

$$\beta_v^\lambda = \langle \beta^\lambda(x), v(x)\rangle_E + \operatorname{div} v(x).$$

Here div is to be understood as the divergence with respect to the Lebesgue measure ℓ on E. On the other hand, taking $\phi_2 = 1$ in the integration by parts formula, we get

$$\operatorname{div}_\lambda v = \beta_v^\lambda.$$

So we obtain the expression

$$\operatorname{div}_\lambda v = \operatorname{div} v + \langle \beta^\lambda, v\rangle_E.$$

Lastly, as in [2,3] we define for any $v \in V_0(E)$,

$$B_v^\lambda(\xi) := \int \langle \beta^\lambda(x), v(x)\rangle_E + \operatorname{div} v(x)\, \xi(\mathrm{d}x). \tag{5.11}$$

Let us now recall the important result from [14], see also [20].

Theorem 5.3.1 (Integration by parts on compacts for determinantal processes)**.** *Assume K, λ satisfy Hypothesis 3, Hypothesis 8, and Hypothesis 9. Let F and G be two cylindrical functions of $\mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda)$. Then, for any compact set $\Lambda \subset E$, we have*

$$\begin{aligned}\int_{\mathcal{X}_\Lambda} \nabla_v F(\xi) G(\xi)\, \mu_{K_\Lambda,\lambda}(d\xi) = &- \int_{\mathcal{X}_\Lambda} F(\xi) \nabla_v G(\xi)\, \mu_{K_\Lambda,\lambda}(d\xi) \\ &+ \int_{\mathcal{X}_\Lambda} F(\xi) G(\xi) (B_v^\lambda(\xi) + \nabla_v U[\Lambda](\xi))\, \mu_{K_\Lambda,\lambda}(d\xi),\end{aligned} \tag{5.12}$$

where B_v^λ is defined in (5.11).

Proof. See [20]. □

Remark 11. *We remark that there is a sign change in (B.3.6), as compared to the results of [14], which is justified by the corrected formula for (5.8). This corrected version is also more in line with the corresponding integration by parts for the Poisson point process.*

In the following, we define the divergence of ∇ with respect to the determinantal measure:

$$\operatorname{div}_v^{K,\lambda} G(\xi) := -\nabla_v G(\xi) + G(\xi) \left(-\beta_v^\lambda(\xi) + \nabla_v U[\Lambda](\xi) \right),$$

for $\xi \in \mathcal{X}_\Lambda$, and $v \in V_0(\Lambda)$.

Next, we extend the operators ∇ and $\operatorname{div}^{K,\lambda}$ by closability. This is given by the following theorem:

Theorem 5.3.2. *Assume that $E := \mathbb{R}^d$ and that the hypotheses of Theorem 5.3.1 hold, as well as the following condition:*

$$\begin{aligned}\int_{\Lambda^n} \left| \frac{\partial_{x_i^{(h)}} \det J[\Lambda](x_1,\dots,x_n) \partial_{x_j^{(k)}} \det J[\Lambda](x_1,\dots,x_n)}{\det J[\Lambda](x_1,\dots,x_n)} \right| \\ \mathbf{1}_{\{\det J[\Lambda](x_1,\dots,x_n)>0\}}\, \lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_n) < \infty\end{aligned} \tag{5.13}$$

for any $n \geq 1$, $1 \leq i, j \leq n$ and $1 \leq h, k \leq d$. Then

(*i*) *the linear operators* ∇_v *and* $\mathrm{div}_v^{K,\lambda}$ *are well-defined and closable for any vector field* $v \in V_0(\Lambda)$. *In particular, we have*

$$\nabla_v(\mathcal{S}_\Lambda) \subset \mathrm{L}^2(\Lambda, \lambda) \qquad \textit{and} \qquad \mathrm{div}_v^{K,\lambda}(\mathcal{S}_\Lambda) \subset \mathrm{L}^2(\Lambda, \lambda);$$

(*ii*) *for any vector field* $v \in V_0(\Lambda)$, *we have*

$$\mathbb{E}\left[\mathbf{G}\overline{\nabla_v}\mathbf{F}\right] = \mathbb{E}\left[\mathbf{F}\overline{\mathrm{div}_v^{K,\lambda}}\mathbf{G}\right]$$

for all $\mathbf{F} \in \mathrm{Dom}\left(\overline{\nabla_v}\right)$, $\mathbf{G} \in \mathrm{Dom}\left(\overline{\mathrm{div}_v^{K,\lambda}}\right)$ *in the domains of the minimal closed extensions of* ∇_v *and* div_v^{μ}. *Here,* $(\overline{\nabla_v}, \mathrm{Dom}(\overline{\nabla_v}))$ *and* $(\overline{\mathrm{div}_v^{K,\lambda}}, \mathrm{Dom}(\overline{\mathrm{div}_v^{K,\lambda}}))$ *are the closed extensions of the respective closable operators* ∇_v *and* $\mathrm{div}_v^{K,\lambda}$ *in the sense of* (5.1).

Proof. Essentially, $\nabla_v(\mathcal{S}_\Lambda) \subset \mathrm{L}^2(\Lambda, \lambda)$ is a consequence of the finiteness of the moments of functionals of determinantal point processes. To ensure $\mathrm{div}_v^{K,\lambda}(\mathcal{S}_\Lambda) \subset \mathrm{L}^2(\Lambda, \lambda)$, we require the additional hypothesis (5.13). Lastly, closability is a consequence of the integration by parts Theorem 5.3.1. Details of the proof can be found in [20]. □

5.3.3 Associated diffusions

In this subsection, we construct the diffusions associated with a determinantal point process satisfying the conditions of Theorem 5.3.2. The interested reader may also see [59, 61, 62] and [38, 39] for constructions of such diffusions by using a similar method. We also refer to [4] for the link between stochastic differential equations and the Dirichlet forms at hand. We begin by studying the Dirichlet form $\mathcal{H}_\Lambda$ defined in (14). First, we show that the integration by parts formula (5.12) leads to the following form for the generator associated to $\mathcal{E}_\Lambda$:

Proposition 5.3.2. *Denote by* $\mathcal{H}_\Lambda$ *the generator associated to* $(\mathcal{E}_\Lambda, \mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda))$ *in the sense of Theorem 5.1.1. Then, for* $F \in \mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda)$, *and* $\xi \in \mathcal{X}_\Lambda$,

$$\mathcal{H}_\Lambda F(\xi) = -\Delta F(\xi) - \langle \mathrm{div}_\lambda \nabla F(\xi), \xi\rangle - \int \langle \nabla_x U[\Lambda], \nabla_x F\rangle_E \, \xi(dx). \tag{5.14}$$

Moreover, $\mathcal{H}_\Lambda$ *is well-defined in such a way as an operator from* $\mathrm{L}^2(\mathcal{X}, \mu)$ *into itself, i.e.*

$$\mathcal{H}_\Lambda(\mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda)) \subset \mathrm{L}^2(\mathcal{X}, \mu). \tag{5.15}$$

Proof. Let F be a function in $\mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda)$. Then recall that

$$\nabla F(\xi) = \sum_{i=1}^N \partial_i f(\int h_1 \mathrm{d}\xi, \ldots, \int h_N \mathrm{d}\xi) \int_E \nabla^E h_i(x)\, \xi(\mathrm{d}x).$$

So,

$$\begin{aligned}\mathrm{div}^\mu(\nabla F)(\xi) = &\sum_{i,j=1}^N \partial_i\partial_j f(\int h_1 \mathrm{d}\xi, \ldots, \int h_N \mathrm{d}\xi) \int \langle \nabla^E h_i, \nabla_x^E h_j(x)\rangle_E \, \xi(\mathrm{d}x) \\ &+ \sum_{i=1}^N \partial_i f(\int h_1 \mathrm{d}\xi, \ldots, \int h_N \mathrm{d}\xi) \langle \mathrm{div}_\lambda \nabla^E h_i + \langle \nabla U[\Lambda](\xi), \nabla^E h_i\rangle_E, \xi\rangle. \end{aligned} \tag{5.16}$$

Then, define

$$\Delta F(\xi) = \mathrm{Tr}(\nabla\nabla F)(\xi) = \sum_{i,j=1}^{N} \partial_i\partial_j f(\int h_1 \mathrm{d}\xi, \dots, \int h_N \mathrm{d}\xi) \int \langle \nabla_x^E h_i, \nabla_x^E h_j \rangle_E \, \xi(\mathrm{d}x),$$

and the identification is complete, once we apply Theorem 5.3.1.

For the proof of (5.15), we refer to the proof of point (i) of Theorem 4.1 in [20]. □

Now, having constructed the generator $\mathcal{H}_\Lambda$ correctly associated to $\mathcal{E}_\Lambda$, Theorem 5.1.1 allows us to conclude to the closability of $\mathcal{E}_\Lambda$. Specifically, we obtain in the same fashion as [2, 3]:

Theorem 5.3.3. *One can define $\mathcal{H}_\Lambda$ which is correctly associated with the pre-Dirichlet form $\mathcal{E}_\Lambda$. As such, the form $(\mathcal{E}_\Lambda, \mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda))$ is closable in $\mathrm{L}^2(\mathcal{X}_\Lambda, \mu_\Lambda)$. Its closure is associated with a positive definite self-adjoint operator (Friedrichs extension of $\mathcal{H}_\Lambda$) which is still noted $\mathcal{H}_\Lambda$. It is defined on the extended domain $\overline{\mathrm{Dom}(\mathcal{E}_\Lambda)}$.*

Proof. This is a consequence of Theorem 5.1.1, see also Theorem 4.1 in [20] for details. □

Recall that $\mathcal{E}_\Lambda$ is associated with a diffusion with (quasi-)continuous sample paths if $\mathcal{E}_\Lambda$ is local and quasi-regular. Therefore, we now wish to verify these conditions. First of all, let us begin by recalling a partial result obtained in [88].

Theorem 5.3.4 (Quasi-regularity and locality, partial result)**.** *Assume that μ is translation-invariant, i.e. $J(x, y) = J(x - y)$, and J is of finite range, i.e. $J(r) = 0$ for $r \geq R$, $R \geq 0$. Then, $(\mathcal{E}_\Lambda, \overline{\mathrm{Dom}(\mathcal{E}_\Lambda)})$ is a local, and quasi-regular form.*

Proof. This was proved in [88]. □

We note that the results obtained in [88] do not strictly extend to our setting since their domain is larger than ours. In the general case, it is possible to verify it by hand (as in [88] or as we have done in [20]).
However, the results in [72] generalize most known results (in particular the ones of [59]). These results are sufficient in our case, and it is known that $\mathcal{E}$ quasi-regular. We obtain:

Proposition 5.3.3. *The Dirichlet form $(\mathcal{E}_\Lambda, \mathrm{Dom}((\mathcal{E}_\Lambda))$ is quasi-regular, in the sense of Definition 11.*

Proof. See our proof in [20]. A shorter proof can be obtained by using Theorem 3.4 in [72] (see also the considerations in Section 4 therein). □

Next, it remains to show that the Dirichlet form is local.

Proposition 5.3.4. *The Dirichlet form $(\mathcal{E}_\Lambda, \mathrm{Dom}((\mathcal{E}_\Lambda))$ is local, in the sense of (5.2).*

Proof. The proof can be found in [20]. □

At this point, we know by Theorem B.5.1 that there exists a diffusion correctly associated with $\mathcal{E}_\Lambda$. Before writing our main result, we obtain further results on the properties of the diffusion. We start by recalling some results from [60]. The Dirichlet form defined there is unchanged, however the domain is $\mathcal{D}_\infty = \{f \in \mathcal{D}_\infty^{\mathrm{loc}} \cap \mathrm{L}^2(\mathcal{X}, \mu) \ : \ \mathcal{E}(f, f) < \infty\}$,

where $\mathcal{D}_\infty^{\mathrm{loc}}$ is the set of functionals such that the restriction of f to configurations of finite size is smooth. We define the set Ξ as follows:

$$\Xi = \{\xi \in \mathcal{N} \ : \ \exists x \in E;\ \xi(\{x\}) \geq 2\},$$

i.e. Ξ is the set of configurations with collisions. In particular, it is known that $\mu(\Xi) = 0$ for any determinantal process μ, since ξ is a.s. a simple point process. The question is whether or not this also holds for the $\mathcal{E}$-capacity of Ξ. The answer is positive:

Proposition 5.3.5 ([60]). *Let μ be a determinantal random point field with kernel K which verifies Hypothesis 3. Assume moreover that K is locally Lipschitz continuous. Then*

$$\mathrm{Cap}_{Osada}(\Xi) = 0,$$

where Cap_{Osada} is the capacity associated with the pre-Dirichlet space $(\mathcal{E}, \mathcal{D}_\infty)$.

In particular, the associated diffusion, constructed on $\mathcal{N}_\Lambda$ has no collisions:

Proposition 5.3.6 ([60]). *Let μ be a determinantal random point field with kernel K which verifies Hypothesis 3. Assume moreover that K is locally Lipschitz continuous. Then the diffusion $(\mathbf{M}_{\mu_\Lambda}, (\theta_t)_{t\geq 0})$ associated with $(\mathcal{E}, \mathcal{D}_\infty)$ satisfies*

$$\mathbf{P}_\eta(\sigma_\Xi = \infty) = 1 \quad \text{for q.e. } \eta \in \ddot{\mathcal{X}},$$

where $\sigma_\Xi = \inf\{t > 0 \ / \ \mathbf{M}_t \in \Xi\}$.

It should be noted that the Dirichlet form defined there is the same as $\mathcal{E}$ but their domain $\mathcal{D}_\infty$ is actually larger than ours. This means that their results do not extend to our setting since $0 = \mathrm{Cap}_{\mathrm{Osada}}(\Xi) \leq \mathrm{Cap}(\Xi)$.
In order to prove the results in our setting, we start by recalling the following lemma, which is borrowed from [73].

Lemma 5.3.4. *Assume the hypotheses of Theorem 5.3.2 and let $(\mathbf{M}_{\mu_\Lambda}, (\theta_t)_{t\geq 0})$ be the μ_Λ-tight special standard process properly associated with the Dirichlet form $(\mathcal{E}, \overline{\mathrm{Dom}(\mathcal{E})})$. Let $u_n(\xi_\Lambda) \in \mathcal{S}_\Lambda$, $n \geq 1$, be such that: $u_n : \mathcal{X}_\Lambda \to \mathbb{R}$ is continuous, $u_n \to u$ point-wise, $u \in \mathrm{L}^2(\Lambda, \lambda)$ and*

$$\sup_{n\geq 1} \mathcal{E}(u_n(\xi_\Lambda), u_n(\xi_\Lambda)) < \infty.$$

Then u is $\overline{\mathcal{E}}$-a.c. and, in particular,

$$\mathbf{P}_\xi(\{\omega : \ t \mapsto u(\mathbf{M}_t)(\omega) \text{ is continuous on } [0, +\infty)\}) = 1, \qquad \mu_\Lambda\text{-a.e. } \xi \in \mathcal{X}_\Lambda.$$

Next theorem provides the non-collision property of $(\mathbf{M}_{\mu_\Lambda}, (\theta_t)_{t\geq 0})$.

Theorem 5.3.5. *Assume here that $E = \mathbb{R}^d$, and $d \geq 2$. Assume moreover that J is continuous on Λ, and that the hypotheses of Theorem 5.3.2 hold. Then $(\mathbf{M}_{\mu_\Lambda}, (\theta_t)_{t\geq 0})$ takes values on $\mathcal{X}_\Lambda$, i.e.*

$$\mathbf{P}_\xi(\{\omega : \ \mathbf{M}_t(\omega) \in \mathcal{X}_\Lambda \quad \forall\, 0 \leq t < \infty\}) = 1, \qquad \mu_\Lambda\text{-a.e. } \xi \in \mathcal{X}_\Lambda.$$

In other words, we have $\mathrm{Cap}(\Xi) = 0$.

Proof. Since the proof is similar to the proof of Proposition 1 in [73] we skip some details. For every positive integer a, define $u := \mathbf{1}_N$, where

$$N := \{\xi \in \mathcal{X}_\Lambda \ : \ \sup_{x\in[-a,a]^d} \xi(\{x\}) \geq 2\}.$$

The claim follows if we prove that u is $\overline{\mathcal{E}}$-a.c.. For this we are going to apply Lemma 5.3.4. Define

$$u_n(\xi) = \Psi\left(\sup_{i\in A_n} \sum_{x\in\mathbf{x}} \phi_i(x)\right), \qquad n \geq 1$$

where $A_n := \mathbb{Z}^d \cap [-na, na]^d$ and Ψ and ϕ are chosen as in the proof of Proposition 1 in [73]. Then $u_n : \mathcal{S}_\Lambda \to \mathbb{R}$ is continuous and $u_n \to u$ point-wise. It remains to prove that $\sup_{n\geq 1} \mathcal{E}(u_n(\xi_\Lambda), u_n(\xi_\Lambda)) < \infty$. For $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, we denote by I_i the function defined by

$$I_i(x) = \prod_{k=1}^d \mathbf{1}_{[-1/2,3/2]}(nx_k - i_k), \qquad x = (x_1, \ldots, x_d) \in \Lambda.$$

The following upper bound holds:

$$\mathcal{E}(u_n(\xi_\Lambda), u_n(\xi_\Lambda)) \leq Cn^2 \sum_{i\in A_n} \mathbb{E}\left[\mathbf{1}_{\left\{\sum_{x\in\xi_\Lambda} I_i(x)\geq 2\right\}} \sum_{x\in\xi_\Lambda} I_i(x)\right], \qquad \text{for some } C > 0.$$

Since μ is stochastically dominated by a Poisson process π^J with intensity measure $J(x,x)\,\lambda(\mathrm{d}x)$ and, for any $i \in A_n$, the mapping $\alpha \mapsto \mathbf{1}_{\left\{\sum_{x\in\alpha} I_i(x)\geq 2\right\}} \sum_{x\in\alpha} I_i(x)$ is "increasing", by (3.21) we have

$$\mathcal{E}(u_n(\xi_\Lambda), u_n(\xi_\Lambda)) \leq Cn^2 \sum_{i\in A_n} \mathbb{E}_{\pi^J}\left[\mathbf{1}_{\left\{\sum_{y\in\xi_\Lambda} I_i(y)\geq 2\right\}} \sum_{y\in\xi_\Lambda} I_i(y)\right], \tag{5.17}$$

where $\mathbb{E}_{\pi^J}$ is the expectation under the Poisson point measure of intensity $J(x,x)\,\lambda(\mathrm{d}x)$. By the properties of the Poisson process, the right hand side of (5.17) is equal to

$$C\,n^2 \sum_{i\in A_n} \left(1 - \mathrm{e}^{-\int_\Lambda I_i(x)J(x,x)\,\lambda(\mathrm{d}x)}\right) \int_\Lambda I_i(x)J(x,x)\,\lambda(\mathrm{d}x),$$

which is bounded above by

$$C\,n^2 \sum_{i\in A_n} \left(\int_\Lambda I_i(x)J(x,x)\rho(x)\,\mathrm{d}x\right)^2.$$

By using the Cauchy-Schwarz inequality, this term is further bounded by

$$C\,n^2 \sum_{i\in A_n} \left(\int_\Lambda I_i(x)\,\mathrm{d}x\right)\left(\int_\Lambda I_i(x)J(x,x)^2\rho(x)^2\,\mathrm{d}x\right). \tag{5.18}$$

We have on the one hand that

$$\int_{\mathbb{R}^d} I_i(x)\,\mathrm{d}x = (2/n)^d$$

and on the other hand, since J and ρ are continuous on the compact set Λ,

$$\int_\Lambda I_i(x)J(x,x)^2\rho(x)^2\,\mathrm{d}x \le \left(\sup_{x\in\Lambda} J(x,x)^2\rho(x)^2\right)(2/n)^d$$

Moreover, $|A_n| \le (2an)^d$. Consequently, the quantity (5.18) is in turn bounded by

$$C_{d,a}\, n^{2-d}, \qquad \text{for some constant } C_{d,a} > 0 \text{ which depends only on } d \text{ and } a.$$

The claim follows by the assumption $d \ge 2$. □

To conclude this subsection, we now give our main result. Here, $\mathbb{E}_\xi$ again denotes the expectation under $\mathbf{P}_\xi$, $\xi \in \mathcal{X}_\Lambda$.

Theorem 5.3.6. *Let $\mu_{\mathcal{K}_\Lambda,\lambda_\Lambda}$ be a determinantal point process on a compact set $\Lambda \subseteq E$ satisfying the conditions of Theorem 5.3.2. Assume moreover that $E = \mathbb{R}^d$, $d \ge 2$. Then there exists a $\mu_{\mathcal{K}_\Lambda,\lambda_\Lambda}$-tight special standard process $(\mathbf{M}_{\mathcal{K}_\Lambda,\lambda_\Lambda}, (\theta_t)_{t\ge 0})$ (we write $\mathbf{M}^\Lambda$ for short) on $\mathcal{X}_\Lambda$ with transition semigroup*

$$p_t f(\xi) := \mathbb{E}_\xi[f(\mathbf{M}_t^\Lambda)], \quad \xi \in \mathcal{X}_\Lambda, \quad f : \mathcal{X}_\Lambda \longrightarrow \mathbb{R} \quad \textit{square integrable.}$$

In addition, $(\mathbf{M}_{\mathcal{K}_\Lambda,\lambda_\Lambda}, (\theta_t)_{t\ge 0})$ is properly associated with the Dirichlet form $(\mathcal{E}_\Lambda, \overline{\mathrm{Dom}(\mathcal{E}_\Lambda)})$ in the sense that $p_t f$ is an $\mathcal{E}_\Lambda$-a.c., $\mu_{\mathcal{K}_\Lambda,\lambda_\Lambda}$-version of $\exp(t\mathcal{H}_\Lambda)f$ for all square integrable $f : \mathcal{X} \longrightarrow \mathbb{R}$ and $t > 0$, and such that

$$\mathbf{P}_\xi(\{\omega :\ t \mapsto \mathbf{M}_t^\Lambda(\omega) \textit{ is continuous on } [0,+\infty)\}) = 1, \qquad \mathcal{E}_\Lambda\text{-a.e., } \xi \in \mathcal{X}_\Lambda, \tag{5.19}$$

i.e. $(\mathbf{M}_{\mathcal{K}_\Lambda,\lambda_\Lambda}, (\theta_t)_{t\ge 0})$ is quasi-continuous.

Proof. The proof is essentially a corollary of Theorem 5.3.5 and Theorem B.5.1, since closability, quasi-regularity and locality were proved previously in this subsection. □

5.3.4 Examples of diffusions

5.3.4.1 Bergman kernel

Let $\Lambda := B(0,R) \subset \mathbb{R}^2$ be the closed ball centered at the origin with radius $R \in (0,1)$, let $\{\varphi_k^{(R)}\}_{1\le k\le N}$, $N \ge 1$, denote the orthonormal subset of $\mathrm{L}^2(B(0,R),\ell)$ defined by

$$\varphi_k^{(R)}(x) := \frac{1}{R}\sqrt{\frac{k+1}{\pi}}\left(\frac{x^{(1)}}{R} + i\frac{x^{(2)}}{R}\right)^k, \quad x = (x^{(1)}, x^{(2)}) \in B(0,R),$$

where $\lambda = \ell$ is the Lebesgue measure on $\mathbb{R}^2$ and $i := \sqrt{-1}$ denotes the complex unit. We consider the modified Bergman operator $\mathcal{K}_{\mathrm{Be}} : B(0,R)^2 \to \mathbb{C}$ with associated kernel defined by

$$K_{\mathrm{Be}}(x,y) := \sum_{k=1}^N R^{2(k+1)}\varphi_k^{(R)}(x)\overline{\varphi_k^{(R)}(y)}, \qquad x,y \in B(0,R), \qquad N \in \mathbb{N}^*$$

and denote by $\mathcal{K}_{\mathrm{Be}}$ the associated integral operator, which is easily seen to be Hermitian and locally of trace class with non-zero eigenvalues $\kappa_k := R^{2(k+1)}$, $k = 1,\dots,N$. As a consequence, the spectrum of $\mathcal{K}_{\mathrm{Be}}$ is contained in $[0,1)$ and the triplet $(\mathcal{K}_{\mathrm{Be}}, K_{\mathrm{Be}}, \ell)$

satisfies assumption (**H1**). In addition, Condition (**H3**) is trivially satisfied since $\lambda = \ell$ is the Lebesgue measure. We note that $\mathcal{K}_{\mathrm{Be}}$ is in fact a restriction operator:

$$\mathcal{K}_{\mathrm{Be}} = \mathcal{P}_{B(0,R)} \mathcal{K'}_{\mathrm{Be}} \mathcal{P}_{B(0,R)},$$

where $\mathcal{K'}_{\mathrm{Be}}$ has kernel

$$K'_{\mathrm{Be}}(x,y) := \sum_{k=1}^{N} \varphi_k^{(1)}(x)\overline{\varphi_k^{(1)}(y)}, \qquad x,y \in B(0,1), \qquad N \in \mathbb{N}^*.$$

Denoting by μ the determinantal process associated to the triplet $(\mathcal{K}'_{\mathrm{Be}}, K'_{\mathrm{Be}}, \ell)$, the Janossy densities of μ on $\Lambda = B(0,R)$ defined in (3.2) are given by

$$j_\Lambda^{(k)}(x_1,\dots,x_k) = \mathrm{Det}(\mathrm{I} - \mathcal{K}_{\mathrm{Be}}) \det J[\Lambda](x_1,\dots,x_k), \qquad k = 1,\dots,N, \quad (x_1,\dots,x_k) \in \Lambda^k,$$

where the kernel $J[\Lambda]$ of $\mathcal{J}[\Lambda]$ is given by

$$J[\Lambda](x,y) := \sum_{h=1}^{N} \frac{R^{2(h+1)}}{1 - R^{2(h+1)}} \varphi_h^{(R)}(x)\overline{\varphi_h^{(R)}(y)},$$

cf. (3.10). Moreover, ξ_Λ has at most N points according to Proposition 3.3.3, which means that $j_\Lambda^k = 0$, for $k \geq N+1$. To prove condition (**H4**) it suffices to remark that the function

$$(x_1,\dots,x_k) \to \det(J[\Lambda](x_p,x_q))_{1\leq p,q\leq k}$$

is continuously differentiable on Λ^k, for $k \leq N$. To show that Condition (5.13) is verified, we first consider the case of j_Λ^N. To that end, note that

$$J[\Lambda](x_1,\dots,x_N) = A^N(x_1,\dots,x_N) A^N(x_1,\dots,x_N)^*,$$

where the matrix $A^N := (A^N_{ph})_{1\leq p,h\leq N}$ is given by

$$A^N_{ph} := \frac{R^{h+1}}{\sqrt{1 - R^{2(h+1)}}} \varphi_h^{(R)}(x_p)$$

and $A^N(x_1,\dots,x_N)^*$ denotes the transpose conjugate of $A^N(x_1,\dots,x_N)$. Hence,

$$\det J[\Lambda](x_1,\dots,x_N) = |\det A^N(x_1,\dots,x_N)|^2,$$

and since the previous determinant is a Vandermonde determinant, we have

$$\det A^N = \prod_{p=1}^{N} \sqrt{\frac{1+p}{\pi(1-R^{2(p+1)})}} \left(\prod_{p=1}^{N} (x_p^{(1)} + i x_p^{(2)}) \right) \prod_{1\leq p<q\leq N} ((x_p^{(1)} - x_q^{(1)}) + i(x_p^{(2)} - x_q^{(2)})). \tag{5.20}$$

Finally, in order to show that Condition (5.13) is satisfied it suffices to check that

$$\int_{B(0,R)^N} \left| \frac{\partial_{x_i^{(p)}} |\det A^N(x_1,\dots,x_N)|^2 \partial_{x_j^{(q)}} |\det A^N(x_1,\dots,x_N)|^2}{|\det A^N(x_1,\dots,x_N)|^2} \right| \mathrm{d}x_1 \dots \mathrm{d}x_N < \infty,$$

for all $1 \leq i, j \leq N$ and $1 \leq p, q \leq 2$, or simply

$$\int_{B(0,R)^N} \left| \frac{\partial_{x_1^{(1)}} |\det A^N(x_1, \ldots, x_N)|^2}{|\det A^N(x_1, \ldots, x_N)|^2} \right| \mathrm{d}x_1 \ldots \mathrm{d}x_N < \infty. \tag{5.21}$$

Now, in order to evaluate the previous quantity, we continue the calculations initiated in (5.20), and obtain

$$\ln |\det A^N|^2 = \ln C_N + \sum_{p=1}^{N} \ln \left((x_p^{(1)})^2 + (x_p^{(2)})^2 \right) + \sum_{1 \leq p < q \leq N} \ln \left((x_p^{(1)} - x_q^{(1)})^2 + (x_p^{(2)} - x_q^{(2)})^2 \right),$$

where $C_N := \prod_{p=1}^{N} \sqrt{\frac{1+p}{\pi(1-R^{2(p+1)})}}$. Hence, the left-hand side of (5.21) reduces to

$$\int_{B(0,R)^N} \left| \frac{2x_1^{(1)}}{(x_1^{(1)})^2 + (x_1^{(2)})^2} + 2 \sum_{j=2}^{N} \frac{x_1^{(1)} - x_j^{(1)}}{(x_1^{(1)} - x_j^{(1)})^2 + (x_1^{(2)} - x_j^{(2)})^2} \right| \mathrm{d}x_1 \ldots \mathrm{d}x_N,$$

which is indeed finite. This proves that Condition (5.13) is verified for $k \geq N$, since it is trivially satisfied for $k > N$. Now, if we take $k < N$, we have again

$$J[\Lambda](x_1, \ldots, x_k) = A^N(x_1, \ldots, x_k) A^N(x_1, \ldots, x_k)^*,$$

where this time, $A^N(x_1, \ldots, x_k)$ is a rectangular $k \times N$ matrix. Hence, by application of the Cauchy-Binet formula:

$$\det J[\Lambda](x_1, \ldots, x_k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} |\det A^{i_1, \ldots, i_k}(x_1, \ldots, x_k)|^2,$$

where we have for $1 \leq p, h \leq k$,

$$A_{ph}^{i_1, \ldots, i_k} := \frac{R^{i_h+1}}{\sqrt{1 - R^{2(i_h+1)}}} \varphi_{i_h}^{(R)}(x_p),$$

which is a square matrix. We now consider fixed $1 \leq i_1 < i_2 < \cdots < i_k \leq N$ and wish to evaluate $|\det A^{i_1, \ldots, i_k}(x_1, \ldots, x_k)|^2$. In fact, we observe that

$$|\det A^{i_1, \ldots, i_k}(x_1, \ldots, x_k)|^2 = \prod_{p=1}^{k} \frac{1 + i_p}{\pi(1 - R^{2(i_p+1)})} \left| V_{i_1, \ldots, i_k}(x_1, \ldots, x_k) \right|^2,$$

where

$$V_{i_1, \ldots, i_k}(x_1, \ldots, x_k) := \det \left(\left(x_h^{i_p} \right)_{1 \leq p, h \leq k} \right)$$

is known in the literature as the generalized Vandermonde determinant. Here, $V_{1, \ldots, k}(x_1, \ldots, x_k)$ is the classical Vandermonde determinant, and in the general case, we have deleted a certain number of rows from the matrix. The generalized Vandermonde determinant is known to factorize into the classical Vandermonde determinant and what is defined to be a Schur polynomial. To be more precise,

$$V_{i_1, \ldots, i_k}(x_1, \ldots, x_k) = V_{1, \ldots, k}(x_1, \ldots, x_k) s_{\lambda(i_1, \ldots, i_k)}(x_1, \ldots, x_k),$$

where $\lambda(i_1,\dots,i_k) := (i_k - k + 1, \dots, i_2 - 1, i_1)$, and s_λ is the Schur polynomial, which is known to be symmetric, and is a sum of monomials, see e.g. [33]. To summarize, we have

$$\det J[\Lambda](x_1,\dots,x_k) = |V_{1,\dots,k}(x_1,\dots,x_k)|^2 \sum_{1\le i_1<i_2<\cdots<i_k\le N} \left(\prod_{p=1}^{k} \frac{1+i_p}{\pi(1-R^{2(i_p+1)})}\right) |s_{\lambda(i_1,\dots,i_k)}(x_1,\dots,x_k)|^2. \quad (5.22)$$

The previous function of $x_1,\dots,x_k$ is of class C^1, therefore condition (**H4**) is verified for $k < N$. Condition (5.13) is verified thanks to the form that s_λ takes, and in particular the fact that $|s_\lambda(x_1,\dots,x_k)|^2 = 0$ if and only if $x_1 = \cdots = x_k = 0$.

We now study the associated diffusion in a simple visual case. Assume that we have $N = 2$, i.e.

$$K_{\mathrm{Be}}(x,y) = \frac{2}{\pi}(x\bar{y}) + \frac{3}{\pi}(x\bar{y})^2,$$

for $x, y \in B(0,R)$. Then, the associated determinantal point process has less than 2 points a.s. and:

$$\nabla_{x_1} U(\{x_1,x_2\}) = -2\left[\frac{x_1}{|x_1|^2} + \frac{x_1-x_2}{|x_1-x_2|^2}\right],$$

for $x_1, x_2 \in \mathbb{R}^2$. Therefore, the diffusion associated with the previous determinantal process starting from $(x_1,x_2) \in \mathbb{C}^2$ is a (weak) solution of the following pair of S.D.E. on $\mathbb{C}$:

$$\begin{cases} \mathrm{d}X_t^{(1)} = \sqrt{2}\,\mathrm{d}B_t^1 + 2\left[\frac{X_t^{(1)}}{|X_t^{(1)}|^2} + \frac{X_t^{(1)}-X_t^{(2)}}{|X_t^{(1)}-X_t^{(2)}|^2}\right]\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = \sqrt{2}\,\mathrm{d}B_t^2 + 2\left[\frac{X_t^{(2)}}{|X_t^{(2)}|^2} + \frac{X_t^{(2)}-X_t^{(1)}}{|X_t^{(2)}-X_t^{(1)}|^2}\right]\mathrm{d}t, \\ X_0^{(1)} = x_1, \quad X_0^{(2)} = x_2, \end{cases}$$

where B^1 and B^2 are two independent complex brownian motions. Moreover, since $s_{\lambda(i)}(x) = x^i$, for $i \in \mathbb{N}$ and $x \in \mathbb{C}$, we have by (5.22)

$$U(\{x\}) = -\ln\left(\frac{2}{\pi(1-R^4)}|x|^2 + \frac{3}{\pi(1-R^6)}|x|^4\right).$$

Therefore, if we set $C_1 := \frac{2}{\pi(1-R^4)}$ and $C_2 := \frac{3}{\pi(1-R^6)}$, then the diffusion starting from $x_1 \in \mathbb{C}$ is a solution of the following S.D.E.:

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t + \frac{2C_1X_t + 4C_2(X_t)^3}{C_1|X_t|^2 + C_2|X_t|^4}\,\mathrm{d}t, \\ X_0 = x_1, \end{cases}$$

where B is a complex brownian motion.

Sine kernel on the circle

Fix $N \in \mathbb{N}^*$ and let $\Lambda := [-\frac{N}{2}, \frac{N}{2}] \subset \mathbb{R}$. Let $\{\varphi_k^{(N)}\}_{0\le k\le N-1}$, denote the orthonormal subset of $\mathrm{L}^2([-\frac{N}{2},\frac{N}{2}],\ell)$ defined by

$$\varphi_k^{(N)}(\theta) := \mathbf{1}_{[-\frac{N}{2},\frac{N}{2}]}(\theta)\frac{1}{\sqrt{N}}e^{2i\pi k\theta/N},$$

where $i := \sqrt{-1}$ denotes the complex unit. We consider the Dyson model on the circle $\mathcal{K}^N_{\mathrm{Dy}} : [-\frac{N}{2}, \frac{N}{2}]^2 \to \mathbb{C}$ with associated kernel defined by

$$K^N_{\mathrm{Dy}}(\theta_1, \theta_2) := \sum_{k=0}^{N-1} \varphi_k^{(N)}(\theta_1)\overline{\varphi_k^{(N)}(\theta_2)}, \qquad \theta_1, \theta_2 \in [-\frac{N}{2}, \frac{N}{2}],$$

i.e. K^N_{Dy} is a projection kernel onto the subspace generated by the family $\{\varphi_k^{(N)}\}_{0\leq k\leq N-1}$. Then, by standard calculations, one can rewrite the previous kernel as

$$K^N_{\mathrm{Dy}}(\theta_1, \theta_2) = \mathbf{1}_{[-\frac{N}{2},\frac{N}{2}]}(\theta_1)\frac{\sin(\pi(\theta_1-\theta_2))}{N\sin(\frac{\pi(\theta_1-\theta_2)}{N})}\mathbf{1}_{[-\frac{N}{2},\frac{N}{2}]}(\theta_2), \qquad \theta_1, \theta_2 \in [-\frac{N}{2}, \frac{N}{2}],$$

and we observe that

$$K^N_{\mathrm{Dy}}(\theta_1, \theta_2) \xrightarrow[N\to\infty]{} \mathrm{sinc}\left(\pi(\theta_1 - \theta_2)\right),$$

uniformly on compact subsets, and where we have written $\mathrm{sinc} := x \mapsto \frac{\sin(x)}{x}$. Therefore by Proposition 3.10 in [77], there is also weak convergence of the determinantal point process with kernel K^N_{Dy} to the one with the sine kernel.

Again, when evaluating the determinant of the matrix $\left(K^N_{\mathrm{Dy}}(\theta_i, \theta_j)\right)_{1\leq i,j\leq N}$, for $\theta_1, \ldots, \theta_N \in [-\frac{N}{2}, \frac{N}{2}]$, we notice that it is nothing but a Vandermonde determinant, which can be explicitly calculated:

$$\begin{aligned}\det\left(K_{\mathrm{Dy}}(\theta_i, \theta_j)\right)_{1\leq i,j\leq N} &= \frac{1}{N^N}\prod_{1\leq i<j\leq N}|e^{2i\pi\theta_j/N} - e^{2i\pi\theta_i/N}|^2 \\ &= \frac{2^{N(N-1)}}{N^N}\prod_{1\leq i<j\leq N}\sin(\pi\frac{\theta_j-\theta_i}{N})^2. \end{aligned} \tag{5.23}$$

Now, even though the operator J is not properly defined, the Janossy density is still well-defined, as was proved in [77]. More precisely, if $\alpha = \{\theta_1, \ldots, \theta_N\}$,

$$j^{(N)}_{[-\frac{N}{2},\frac{N}{2}]}(\alpha) = \det\left(K_{\mathrm{Dy}}(\theta_i, \theta_j)\right)_{1\leq i,j\leq N},$$

and if $|\alpha| \neq N$, then $j_{[-\frac{N}{2},\frac{N}{2}]}(\alpha) = 0$. As before, condition (**H4**) is satisfied as the previous function is continuously differentiable on $[-\frac{N}{2}, \frac{N}{2}]$. Moreover, thanks to (5.23), we obtain

$$\begin{aligned}&\left|\frac{\partial_{\theta_i} j^{(N)}_{[-\frac{N}{2},\frac{N}{2}]}(\theta_1, \ldots, \theta_n)\partial_{\theta_j} j^{(N)}_{[-\frac{N}{2},\frac{N}{2}]}(\theta_1, \ldots, \theta_n)}{j^{(N)}_{[-\frac{N}{2},\frac{N}{2}]}(\theta_1, \ldots, \theta_n)}\right| \\ &\qquad = C\prod_{k<k'}\sin\left(\pi\frac{\theta_k-\theta_{k'}}{N}\right)^2\left|\sum_{k\neq i,k'\neq j}\mathrm{cotan}\left(\pi\frac{\theta_i-\theta_k}{N}\right)\mathrm{cotan}\left(\pi\frac{\theta_j-\theta_{k'}}{N}\right)\right|,\end{aligned}$$

for $i, j = 1, \ldots, N$, and $\theta_1, \ldots, \theta_N \in [-\frac{N}{2}, \frac{N}{2}]$. The previous equation being continuous in $\theta_1, \ldots, \theta_N$, it is integrable on a compact, and therefore Condition (5.13) is verified.

Therefore, the diffusion associated with the previous determinantal process starting from $(\theta_1, \dots, \theta_N) \in [-\frac{N}{2}, \frac{N}{2}]^N$ is a (weak) solution of the following system of S.D.E. on $\mathbb{R}$:

$$\begin{cases} \mathrm{d}\theta_t^{(1)} = \sqrt{2}\,\mathrm{d}B_t^1 + \sum_{1\le j\le N\ j\neq i} \frac{2\pi}{N} \cot\left(\frac{\pi(\theta_t^{(j)} - \theta_t^{(1)})}{N}\right) \mathrm{d}t, \\ \vdots \\ \mathrm{d}\theta_t^{(N)} = \sqrt{2}\,\mathrm{d}B_t^N + \sum_{1\le j\le N\ j\neq i} \frac{2\pi}{N} \cot\left(\frac{\pi(\theta_t^{(j)} - \theta_t^{(N)})}{N}\right) \mathrm{d}t, \\ \theta_0^{(1)} = \theta_1, \dots, \theta_0^{(N)} = \theta_N, \end{cases} \tag{5.24}$$

where $B^1, \dots, B^N$ are N independent brownian motions on $\mathbb{R}$. It can be noted that the S.D.E. (5.24) was solved in the strong sense in [15].

5.3.5 Generalization of the integration by parts formula to the whole space

In this subsection, we generalize Theorem 5.3.2 to the whole space. We begin by introducing an increasing series of compact sets $(\Lambda^n)_{n\in\mathbb{N}}$, such that $\cup_{n\in\mathbb{N}}\Lambda^n = E$. For such $(\Lambda^n)_{n\in\mathbb{N}}$, we denote by

$$\mathcal{E}_n(F, G) := \mathcal{E}_{\Lambda^n}(F, G),$$

for $F, G \in \mathcal{FC}_b^\infty(C^\infty(\Lambda^n), \mathcal{X}_{\Lambda^n}) := \mathrm{Dom}(\mathcal{E}_n)$. We recall that in Subsection 5.3.3, we showed that $(\mathcal{E}_n, \mathrm{Dom}(\mathcal{E}_n))$ is a closable quasi-regular local Dirichlet form, which is associated with $\mathcal{H}_n := \mathcal{H}_{\Lambda^n}$. Then, we obtain

Theorem 5.3.7 (Dirichlet form on E)**.** *Let $\mu_{\mathcal{K},\lambda}$ be a determinantal point process satisfying the hypotheses of Theorem 5.3.2. Assume moreover that the operator $\mathcal{J}$ associated with the determinantal point process is continuous. Define $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ as in (5.6). Then, $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a closable, quasi-regular, and local Dirichlet form.*

Proof. We begin by remarking that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a densely defined (since it contains at least $\mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X})$) positive definite symmetric bilinear form. It is well-defined by Proposition 5.2.2, since $\mu_{\mathcal{K},\lambda}$ has finite moments.
We remark that $(\mathcal{E}_n, \mathrm{Dom}(\mathcal{E}_n))$ are increasing in the sense of (5.4). Therefore, by applying Lemma 5.1.2, we obtain that $\mathcal{E}$ is closable as it is the increasing limit of $(\mathcal{E}_n, \mathrm{Dom}(\mathcal{E}_n))$. Its domain is defined in Lemma 5.1.2 as $\mathrm{Dom}(\mathcal{E}) := \mathcal{D}^\infty := \{f \in \cap_{n\in\mathbb{N}}\mathrm{Dom}(\mathcal{E}_n) : \sup_n \mathcal{E}_n(f, f) < \infty\}$. In the following, we write $(\mathcal{E}, \overline{\mathcal{D}^\infty})$ for its closure.
The Dirichlet property follows by using the arguments in [20]. It is in fact an intrinsic property of the gradient ∇ on cylindrical functions of $\mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X})$. Lastly, in order to verify that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is quasi-regular and local, it suffices to apply Theorem 5.1.7. Indeed, we have already proved **(A1)** and **(A2)** follows since by the results of [28], $j_r^i(x_1, \dots, x_i) \le J(x_1, x_1) \dots J(x_i, x_i)$ for λ-a.e. $x_1, \dots, x_i \in \mathcal{X}_{B(0,r)}$. We conclude that $j_r^i(x_1, \dots, x_i) \in \mathrm{L}^\infty(\mathcal{X}_r^i, \mathrm{d}\lambda)$ by continuity of J. The last part of **(A2)** follows directly by the fact that $\mathcal{K}$ is locally trace-class. Hence, $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a closable quasi-regular local Dirichlet form. □

As a corollary, we obtain

Corollary 5.3.1. *Assume that the hypotheses of Theorem 5.3.7 hold. Then, there exists a nonpositive definite self-adjoint operator, denoted by $-\mathcal{H}$ such that*

$$\mathbb{E}[(-\mathcal{H}F)\,G] = \mathbb{E}[\int \langle \nabla_x F(\xi), \nabla_x G(\xi)\rangle_E\, \xi(dx)], \tag{5.25}$$

for $F, G \in \overline{\mathrm{Dom}(\mathcal{E})}$.

Proof. This is a direct consequence of Theorem 5.3.7 and Theorem 5.1.1 □

Remark 12. *When we take a function* F *in* $\mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X})$*, we remark that* (5.25) *is an equivalent formulation of Theorem 5.3.2 for* μ.

And lastly, there exists a quasi-continuous diffusion correctly associated with the determinantal point process on $\mathcal{X}$.

Theorem 5.3.8. *Let* $\mu_{\mathcal{K},\lambda}$ *be a determinantal point process satisfying the conditions of Theorem 5.3.2. We assume moreover that* $d \geq 2$ *and* $\rho(\cdot)J(\cdot,\cdot) \in \mathrm{L}^2_{\mathrm{loc}}$*. Then there exists a* $\mu_{\mathcal{K},\lambda}$*-tight special standard process* $(\mathbf{M}_{\mathcal{K},\lambda}, (\theta_t)_{t\geq 0})$ *on* $\mathcal{X}$ *with transition semigroup*

$$p_t f(\xi) := \mathbb{E}_\xi[f(\mathbf{M}_t)], \quad \xi \in \mathcal{X}, \quad f : \mathcal{X} \longrightarrow \mathbb{R} \quad \text{square integrable.}$$

In addition, $(\mathbf{M}_{\mathcal{K},\lambda}, (\theta_t)_{t\geq 0})$ *is properly associated with the Dirichlet form* $(\mathcal{E}, \overline{\mathrm{Dom}(\mathcal{E})})$ *in the sense that* $p_t f$ *is an* $\mathcal{E}$*-a.c.,* $\mu_{\mathcal{K},\lambda}$*-version of* $\exp(t\mathcal{H})f$ *for all square integrable* $f : \mathcal{X} \longrightarrow \mathbb{R}$ *and* $t > 0$*, and such that*

$$\mathbf{P}_\xi(\{\omega : \ t \mapsto \mathbf{M}_t(\omega) \text{ is continuous on } [0, +\infty)\}) = 1, \qquad \mathcal{E}\text{-a.e., } \xi \in \mathcal{X}, \tag{5.26}$$

i.e. $(\mathbf{M}_{\mathcal{K},\lambda}, (\theta_t)_{t\geq 0})$ *is quasi-continuous. Moreover, such* $\mathbf{M}$ *is up to* $\mu_{\mathcal{K},\lambda}$*-equivalence unique (in the sense of [49]) and hence* $\mu_{\mathcal{K},\lambda}$ *is reversible with respect to* $(\mathbf{M}_{\mathcal{K},\lambda}, (\theta_t)_{t\geq 0})$.

Proof. The first part is a consequence of Theorem 5.3.7. A finer study of the proof of Theorem 5.3.5 shows moreover that the associated diffusion $(\mathbf{M}_{\mathcal{K},\lambda}, (\theta_t)_{t\geq 0})$ has no collisions by adding the hypothesis $d \geq 2$ and $\rho(\cdot)J(\cdot,\cdot) \in \mathrm{L}^2_{\mathrm{loc}}$. Lastly, reversibility of $\mu_{\mathcal{K},\lambda}$ follows from Theorem 6.4 page 141 in [49]. □

5.4 General point process

The ideas developed in this section are inspired from the work of [46]. Here, we go one step further than what is done there, and give sufficient conditions for the existence of an integration by parts formula for a general point processes satisfying Hypothesis 4 (Σ_λ) (i.e. $C_\mu \ll \lambda \otimes \mu$). These conditions are in particular satisfied for all known integration by parts formulae, which leads us to believe that it is indeed a relevant condition.

Assuming that an integration by parts formula of the type (5.10) holds on E, we exhibit conditions ensuring an integration by parts on $\mathcal{X}$ in the following theorem:

Theorem 5.4.1. *Assume that a given point process* μ *satisfies Hypothesis 4. Assume that* $\sqrt{\rho} \in H^{1,2}_{\mathrm{loc}}$ *(this is a weaker version of Hypothesis 8). Assume moreover that its Papangelou intensity* c *satisfies, for* μ*-almost all* $\xi \in \mathcal{X}$, $\sqrt{c(\cdot,\xi)} \in H^{1,2}_{\mathrm{loc}}$*. Here,* $H^{1,2}_{\mathrm{loc}}$ *denotes the local Sobolev space of order* 1 *on* (E, λ)*, i.e. the space*

$$\{u : E \to \mathbb{R} \ : \ \forall A \subseteq E, \quad \int_A u(x)^2 \, \lambda(dx) < \infty \text{ and } \int_A \|\nabla_x u(x)\|^2 \, \lambda(dx) < \infty\}. \tag{5.27}$$

Then μ *satisfies an integration by parts formula for all* $F, G \in \mathcal{FC}_b^\infty(C^\infty(E), \mathcal{X})$*, for all* $v \in V_0(E)$*,*

$$\int F \nabla_v G \, d\mu = -\int G \nabla_v F \, d\mu - \int GF B_v^c \, d\mu,$$

where

$$B_v^c : \xi \mapsto \int (\langle \frac{\nabla^E c}{c}(x, \xi \setminus x), v(x)\rangle_E + \langle \frac{\nabla^E \rho}{\rho}(x), v(x)\rangle_E + \operatorname{div} v(x))\, \xi(dx).$$

Proof. Let us recall that for $\phi, \psi \in C_0^\infty(E)$, for a measure λ on $(E, \mathcal{B})$ with density $\mathrm{d}\lambda = \rho\, \mathrm{d}\ell$ satisfying $\sqrt{\rho} \in H^{1,2}_{\mathrm{loc}}$, as seen in (5.10), the following integration by parts holds

$$\int \phi \nabla_v^E \psi \, \mathrm{d}\lambda = -\int \psi \nabla_v^E \phi \, \mathrm{d}\lambda - \int \phi\psi\beta_v^\lambda \, \mathrm{d}\lambda,$$

for compactly supported vector fields $v \in V_0(E)$, and where

$$\beta_v^\lambda : x \mapsto \langle \frac{\nabla^E \rho}{\rho}(x), v(x)\rangle_E + \operatorname{div} v(x).$$

Here, div is the divergence with respect to the Lebesgue measure ℓ on E. For details on this as well as why we choose $\sqrt{\rho} \in H^{1,2}_{\mathrm{loc}}$, see [7]. Then, for $F, G \in \mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})$,

$$\begin{aligned}
\int F \nabla_v G \, \mathrm{d}\mu &= \int \int F(\xi) \langle \nabla_x G(\xi), v(x)\rangle_E \, \xi(\mathrm{d}x)\, \mu(\mathrm{d}\xi) \\
&= \int \int F(\xi) \langle \nabla^E D_x G(\xi \setminus x), v(x)\rangle_E \, \xi(\mathrm{d}x)\, \mu(\mathrm{d}\xi) \\
&= \int \int F(\xi \cup x) \langle \nabla^E D_x G(\xi), v(x)\rangle_E \, c(x, \xi)\, \rho(x)\, \ell(\mathrm{d}x)\, \mu(\mathrm{d}\xi) \\
&= -\int \int G(\xi \cup x) \langle \nabla^E D_x F(\xi), v(x)\rangle_E \, \ell(\mathrm{d}x)\, \mu(\mathrm{d}\xi) \\
&\quad - \int \int F(\xi \cup x) G(\xi \cup x) \beta_v^c(x, \xi)\, \ell(\mathrm{d}x)\, \mu(\mathrm{d}\xi),
\end{aligned}$$

where we have used the relation between ∇^E and ∇_x which traces back to [65] to the best of our knowledge. Here, β_v^c is the logarithmic derivative associated with the measure $c(x, \xi)\rho(x)\ell(\mathrm{d}x)$. More precisely,

$$\beta_v^c : (x, \xi) \mapsto \langle \frac{\nabla^E c}{c}(x, \xi), v(x)\rangle_E + \langle \frac{\nabla^E \rho}{\rho}(x), v(x)\rangle_E + \operatorname{div} v(x),$$

which leads to the integration by parts on $\mathcal{X}$:

$$\int F \nabla_v G \, \mathrm{d}\mu = -\int G \nabla_v F \, \mathrm{d}\mu - \int F(\xi) G(\xi) \int \beta_v^c(x, \xi \setminus x)\, \xi(\mathrm{d}x)\, \mu(\mathrm{d}\xi).$$

□

Example 1. *The Poisson measure corresponds to $c(x, \xi) = z(x)$, $x \in E$, $\xi \in \mathcal{X}$.*

Example 2. *Let us consider the integration by parts formula for the determinantal process on compact sets. Thus, consider μ_Λ with Papangelou intensity $c_\Lambda(x, \xi) := \frac{\det J[\Lambda](\xi \cup x)}{\det J[\Lambda](\xi)}$, as seen in (3.18). Then, under the assumptions of [14],*

$$\frac{\nabla^E c_\Lambda}{c_\Lambda}(x, \xi \setminus x) = -\nabla^E D_x U[\Lambda](\xi \setminus x) = -\nabla_x U[\Lambda](\xi), \tag{5.28}$$

and hence, if the integration by parts holds, it is necessarily the same as the one in [14, 20].

In the next theorem, we prove that the integrability condition holds, and therefore the integration by parts formula for determinantal point processes is obtained as a corollary of Theorem 5.4.1.

Theorem 5.4.2. *Let Λ be a compact set of $E := \mathbb{R}^d$. Consider the measure μ_Λ and its associated Papangelou intensity $c_\Lambda(x,\xi) = \frac{\det J[\Lambda](\xi \cup x)}{\det J[\Lambda](\xi)}$ (obtained in (3.18)). Assume that $J[\Lambda]$ is once differentiable with continuous gradient. Assume moreover that (5.13) is satisfied. Then, for μ-almost all $\xi \in \mathcal{X}$, $\sqrt{c_\Lambda(\cdot,\xi)} \in H^{1,2}_{\mathrm{loc}}$.*

Proof. Consider a compact set $A \subseteq E$, then

$$\mathbb{E}[\int_A |c_\Lambda(x,\xi)|\,\mathrm{d}x] \leq \mathbb{E}[\int_A J[\Lambda](x,x)\,\mathrm{d}x] < \infty,$$

by (3.19) and since the operator $J[\Lambda]$ is trace-class. To verify the second condition of (5.27), we remark that by (5.13), we have that

$$\mathbb{E}[\int_A \|\nabla_x U[\Lambda](\xi)\|^2\,\xi(\mathrm{d}x)] < \infty.$$

Then, by (5.28), we have

$$\begin{aligned}\mathbb{E}[\int_A \|\nabla_x U[\Lambda](\xi)\|^2\,\xi(\mathrm{d}x)] &= \mathbb{E}[\int_A \|\nabla^E_x \ln c_\Lambda(x,\xi\setminus x)\|^2\,\xi(\mathrm{d}x)] \\ &= \int_{A\times\mathcal{X}} \|\nabla^E_x \ln c_\Lambda(x,\xi)\|^2\,C_{\mu_\Lambda}(\mathrm{d}x,\mathrm{d}\xi),\end{aligned}$$

where C_{μ_Λ} is the Campbell measure of μ_Λ defined in (3.12). Now, remark that by the same arguments as those of Lemma 5.3.3, we have

$$C_{\mu_\Lambda}(\{c_\Lambda(x,\xi)=0\}) = \int_{\Lambda\times\mathcal{X}} \mathbf{1}_{\{c_\Lambda(x,\xi)=0\}} c_\Lambda(x,\xi)\,\lambda(\mathrm{d}x)\,\mu(\mathrm{d}\xi) = 0,$$

since c_Λ is the density of C_{μ_Λ} with respect to $\lambda\otimes\mu_\Lambda$. Hence,

$$\begin{aligned}\mathbb{E}[\int_A \|\nabla_x U[\Lambda](\xi)\|^2\,\xi(\mathrm{d}x)] &= \int_{A\times\mathcal{X}} \|\nabla^E_x \ln c_\Lambda(x,\xi)\|^2 \mathbf{1}_{\{c_\Lambda(x,\xi)>0\}}\,C_{\mu_\Lambda}(\mathrm{d}x,\mathrm{d}\xi) \\ &= \int_{A\times\mathcal{X}} \|\nabla^E_x \ln c_\Lambda(x,\xi)\|^2 \mathbf{1}_{\{c_\Lambda(x,\xi)>0\}}\,C_{\mu_\Lambda}(\mathrm{d}x,\mathrm{d}\xi) \\ &= \int_{A\times\mathcal{X}} \|\nabla^E_x \ln c_\Lambda(x,\xi)\|^2 \mathbf{1}_{\{c_\Lambda(x,\xi)>0\}} c_\Lambda(x,\xi)\,\lambda(\mathrm{d}x)\,\mu(\mathrm{d}\xi) \\ &= \int_{A\times\mathcal{X}} \frac{1}{c_\Lambda(x,\xi)}\|\nabla^E_x c_\Lambda(x,\xi)\|^2 \mathbf{1}_{\{c_\Lambda(x,\xi)>0\}}\,\lambda(\mathrm{d}x)\,\mu(\mathrm{d}\xi) \\ &= \int_{A\times\mathcal{X}} 4\|\nabla^E_x \sqrt{c_\Lambda(x,\xi)}\|^2 \mathbf{1}_{\{c_\Lambda(x,\xi)>0\}}\,\lambda(\mathrm{d}x)\,\mu(\mathrm{d}\xi),\end{aligned}$$

and we conclude by noticing that a finer study of the integration by parts on E shows that it suffices to show that $\sqrt{c_\Lambda(\cdot,\xi)}\mathbf{1}_{\{c_\Lambda(\cdot,\xi)>0\}} \in H^{1,2}_{\mathrm{loc}}$. □

The previous approach presents the advantage of relying only on properties of the Papangelou intensity in order to obtain an integration by parts formula. Hence, we are able to fully incorporate the deeper results of [28] which were recalled in Theorem 3.4.3. More specifically, we obtain this generalized version of Theorem 5.3.2:

Theorem 5.4.3. *Consider the measure $\mu_{\mathcal{K},\lambda}$ which verifies Hypothesis 3 and its associated Papangelou intensity defined for almost all $(x,\xi) \in E \times \mathcal{X}$ by $c(x,\xi) = \lim_n c_{\Lambda_n}(x,\xi_{\Lambda_n})$. Assume that Hypothesis 5 holds and that (5.13) is satisfied for the operator J instead of $J[\Lambda]$. Then, for almost all $\xi \in \mathcal{X}$, $\sqrt{c(\cdot,\xi)} \in H^{1,2}_{\mathrm{loc}}$. In that case, the following integration by parts holds, for all $F, G \in \mathcal{FC}^\infty_b(C^\infty(E),\mathcal{X})$, for all $v \in V(E)$,*

$$\int F\nabla_v G \, d\mu = -\int G\nabla_v F \, d\mu - \int FG\left(\nabla_v U(\xi) + \langle \mathrm{div}_\lambda v, \xi\rangle\right) \mu(d\xi), \tag{5.29}$$

where we have defined the operator $U(\xi) := -\ln\det J(\xi)$, for $\xi \in \mathcal{X}$.

Proof. Here, we know from equation (3.9) in [28] that $c(x,\xi) = \lim_n \frac{\det J(x\cup\xi_{\Lambda^n})}{\det J(\xi_{\Lambda^n})}$, for any sequence of compact sets $(\Lambda^n)_{n\in\mathbb{N}}$ increasing to E. Moreover, given the previous conditions, the previous sequence is stationary $\lambda\otimes\mu$-everywhere. More precisely, if we note $W(x,\xi)$ the union of clusters hitting x, we have $c(x,\xi) = \frac{\det J(x\cup\xi_{W(x,\xi)})}{\det J(\xi_{W(x,\xi)})}\mathbf{1}_{\mathrm{diam}\, W(x,\xi)<\infty}$, and $\mu(\mathrm{diam}\, W(x,\xi) < \infty) = 1$ by Hypothesis 5. Then, we proceed as previously, and consider a compact set $A \subseteq E$, and obtain

$$\mathbb{E}[\int_A |c(x,\xi)|\,\mathrm{d}x] \leq \mathbb{E}[\int_A J(x,x)\,\mathrm{d}x] < \infty,$$

by (3.19) and since the operator J is locally trace-class. To verify the second condition of (5.27), we remark that by (5.13), we have that

$$\mathbb{E}[\int_A \|\nabla_x U(\xi)\|^2\,\xi(\mathrm{d}x)] < \infty.$$

Then, by (5.28), we have

$$\begin{aligned}\mathbb{E}[\int_A \|\nabla_x U(\xi)\|^2\,\xi(\mathrm{d}x)] = \mathbb{E}[\int_A \|\nabla^E_x \ln c(x,\xi\setminus x)\|^2\,\xi(\mathrm{d}x)] \\ = \int_{A\times\mathcal{X}} \|\nabla^E_x \ln c(x,\xi)\|^2\, C_\mu(\mathrm{d}x,\mathrm{d}\xi),\end{aligned}$$

where C_μ is the Campbell measure of μ defined in (3.12). Now, as before,

$$C_\mu(\{c(x,\xi) = 0\}) = 0.$$

Hence, the calculations are precisely the same as those of the previous theorem:

$$\mathbb{E}[\int_A \|\nabla_x U(\xi)\|^2\,\xi(\mathrm{d}x)] = \int_{A\times\mathcal{X}} 4\|\nabla^E_x\sqrt{c(x,\xi)}\|^2\mathbf{1}_{\{c(x,\xi)>0\}}\,\lambda(\mathrm{d}x)\,\mu(\mathrm{d}\xi),$$

and we conclude that $\sqrt{c(\cdot,\xi)} \in H^{1,2}_{\mathrm{loc}}$. Hence, by applying Theorem 5.4.1, we obtain (5.29). □

We remark that once we obtain Theorem 5.4.3, we can also generalize the domain of F, G to the extended domain of $\overline{\nabla}$. It is also possible to show closability of the associated Dirichlet operator, and the existence of diffusions.

More specifically, we prove the previous consideration in a general setting satisfied in particular in Theorem 5.4.3. The main additional hypothesis is the following:

Hypothesis 10. *We assume that μ satisfies Hypothesis 4 (Σ_λ). Moreover, its Papangelou intensity c is differentiable in its first variable and satisfies for all compact sets $A \subseteq E$,*

$$\int_{A\times\mathcal{X}} (1+\xi(A))\|\nabla^E \ln c(x,\xi)\|^2 \, C_\mu(dx, d\xi) < \infty.$$

We remark that Hypothesis 10 is stronger than $\sqrt{c(\cdot,\xi)} \in H^{1,2}_{\mathrm{loc}}$ for μ-a.e. $\xi \in \mathcal{X}$. Indeed, it implies that $\int_{A\times\mathcal{X}} \|\nabla^E_x \ln c(x,\xi)\|^2 \, C_\mu(\mathrm{d}x, \mathrm{d}\xi) < \infty$ and moreover,

$$\begin{aligned}
\int_{A\times\mathcal{X}} \|\nabla^E_x \ln c(x,\xi)\|^2 \, C_\mu(\mathrm{d}x, \mathrm{d}\xi) &= \int_{A\times\mathcal{X}} \|\nabla^E_x \ln c(x,\xi)\|^2 \, C_\mu(\mathrm{d}x, \mathrm{d}\xi) \\
&= \int_{A\times\mathcal{X}} \|\nabla^E_x \ln c(x,\xi)\|^2 c(x,\xi) \, \lambda(\mathrm{d}x)\, \mu(\mathrm{d}\xi) \\
&= \int_{A\times\mathcal{X}} \frac{1}{c(x,\xi)} \|\nabla^E_x c(x,\xi)\|^2 \, \lambda(\mathrm{d}x)\, \mu(\mathrm{d}\xi) \\
&= \int_{A\times\mathcal{X}} 4\|\nabla^E_x \sqrt{c(x,\xi)}\|^2 \, \lambda(\mathrm{d}x)\, \mu(\mathrm{d}\xi),
\end{aligned}$$

which implies that $\nabla^E \sqrt{c(\cdot,\xi)} \in \mathrm{L}^2_{\mathrm{loc}}(E,\lambda)$ for μ-a.e. $\xi \in \mathcal{X}$. Moreover, the first part of (5.27) always holds, since for a compact set $A \subseteq E$, we have by (3.16):

$$\mathbb{E}[\int_A c(x,\xi)\, \lambda(\mathrm{d}x)] = \int_A \rho(x)\, \lambda(\mathrm{d}x) = \mathbb{E}[\xi(A)] < \infty.$$

Then, under Hypothesis 10, by a straightforward application of Theorem 5.4.1, μ satisfies an integration by parts formula for all $F, G \in \mathcal{F}C^\infty_b(C^\infty(E), \mathcal{X})$, for all $v \in V_0(E)$,

$$\int F \nabla_v G \, \mathrm{d}\mu = -\int G \nabla_v F \, d\mu - \int FG B^c_v \, \mathrm{d}\mu,$$

where

$$B^c_v : \xi \mapsto \int (\langle \frac{\nabla^E c}{c}(x, \xi\setminus x), v(x)\rangle_E + \operatorname{div} v(x))\, \xi(\mathrm{d}x).$$

Therefore, we are now able to proceed exactly as in [20] and as we did in this manuscript in Section 5.3. In order to conserve similar notations, we can simply set

$$\operatorname{div}^\mu_v F(\xi) := -\nabla_v F(\xi) - F(\xi) B^c_v(\xi), \quad \forall \xi \in \mathcal{X},$$

and obtain the following analogue of Theorem 5.3.2.

Theorem 5.4.4. *Assume that $E := \mathbb{R}^d$ and that the hypotheses of Theorem 5.4.1 hold, as well as Hypothesis 10. Then*

(i) the linear operators ∇_v and div^μ_v are well-defined and closable for any vector field $v \in V_0(E)$. In particular, we have

$$\nabla_v(\mathcal{F}C^\infty_b(C^\infty(E),\mathcal{X})) \subset \mathrm{L}^2(\mathcal{X},\mu) \qquad \text{and} \qquad \operatorname{div}^\mu_v(\mathcal{F}C^\infty_b(C^\infty(E),\mathcal{X})) \subset \mathrm{L}^2(\mathcal{X},\mu),$$

(ii) for any vector field $v \in V_0(E)$, we have

$$\mathbb{E}\left[\mathbf{G}\overline{\nabla_v}\mathbf{F}\right] = \mathbb{E}\left[\mathbf{F}\overline{\operatorname{div}^\mu_v}\mathbf{G}\right]$$

for all $\mathbf{F} \in \operatorname{Dom}\left(\overline{\nabla_v}\right)$, $\mathbf{G} \in \operatorname{Dom}\left(\overline{\operatorname{div}^\mu_v}\right)$ in the domains of the minimal closed extensions of ∇_v and div^μ_v. Here, $(\overline{\nabla_v}, \operatorname{Dom}(\overline{\nabla_v}))$ and $(\overline{\operatorname{div}^\mu_v}, \operatorname{Dom}(\overline{\operatorname{div}^\mu_v}))$ are the closed extensions of the respective closable operators ∇_v and div^μ_v in the sense of (5.1).

Proof. (i) The operator ∇_v is not dependent on the underlying measure μ. Hence, ∇_v is still well-defined and closable. We now wish to prove that div_v^μ is well-defined (closability follows in exactly the same way as in [20]). Recall that

$$\mathrm{div}_v^\mu F(\xi) := -\nabla_v F(\xi) - F(\xi)\int (\langle \frac{\nabla^E c}{c}(x, \xi \setminus x), v(x)\rangle_E + \mathrm{div}\, v(x))\, \xi(\mathrm{d}x),$$

for $\xi \in \mathcal{X}$. Since for $F \in \mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})$, $\nabla_v F(\xi) \in \mathrm{L}^2(\mathcal{X}, \mu)$, and since v is compactly supported, it remains to prove that

$$\int \langle \frac{\nabla^E c}{c}(x, \xi \setminus x), v(x)\rangle_E\, \xi(\mathrm{d}x) \in \mathrm{L}^2(\mathcal{X}, \mu).$$

This follows easily from Hypothesis 10:

$$\begin{aligned}
&\mathbb{E}[(\int \langle \frac{\nabla^E c}{c}(x, \xi \setminus x), v(x)\rangle_E\, \xi(\mathrm{d}x))^2] \\
&\qquad \leq \mathbb{E}[\xi(A) \int \langle \frac{\nabla^E c}{c}(x, \xi \setminus x), v(x)\rangle_E^2\, \xi(\mathrm{d}x)] \\
&\qquad = \int_{A\times\mathcal{X}} (1 + \xi(A))\langle \frac{\nabla^E c}{c}(x, \xi), v(x)\rangle_E^2 C_\mu(\mathrm{d}x, \mathrm{d}\xi) \\
&\qquad \leq \sup_{x\in A} \|v(x)\|^2 \int_{A\times\mathcal{X}} (1 + \xi(A))\|\frac{\nabla^E c}{c}(x, \xi)\|^2 C_\mu(\mathrm{d}x, \mathrm{d}\xi) \\
&\qquad < \infty,
\end{aligned}$$

where A is the support of v.

(ii) The proof of (ii) does not depend on specific properties of div_v^μ, therefore the proof is as in [20].

□

As before, we define for $F, G \in \mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})$, the following positive definite symmetric form

$$\mathcal{E}(F, G) = \frac{1}{2}\int_{\mathcal{X}} \int \langle \nabla_x F, \nabla_x G\rangle\, \xi(\mathrm{d}x)\, \mu(\mathrm{d}\xi), \tag{5.30}$$

with domain $\mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X}) \subset \mathrm{L}^2(\mathcal{X}, \mu)$. We note in particular that $\mathcal{E}$ is well-defined since $\sqrt{c}(\cdot, \xi) \in H^{1,2}_{\mathrm{loc}}$ implies in particular that μ has finite moments, and therefore Proposition 5.2.2 can be applied. We now study the generator $\mathcal{H}$ associated with the Dirichlet form $\mathcal{E}$.

Proposition 5.4.1. *Denote by $\mathcal{H}$ the generator associated to $(\mathcal{E}, \mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda))$ in the sense of Theorem 5.1.1. Then, for $F \in \mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})$, and $\xi \in \mathcal{X}$,*

$$\mathcal{H}F(\xi) = -\mathrm{div}^\mu \nabla F(\xi) = -\Delta F(\xi) - \langle \mathrm{div}^\mu \nabla F(\xi), \xi\rangle - \int \langle \nabla^E \ln c(x, \xi \setminus x), \nabla F\rangle_E\, \xi(dx). \tag{5.31}$$

Moreover, $\mathcal{H}$ is well-defined in such a way as an operator from $\mathrm{L}^2(\mathcal{X}, \mu)$ into itself, i.e.

$$\mathcal{H}(\mathcal{F}C_b^\infty(C^\infty(E), \mathcal{X})) \subset \mathrm{L}^2(\mathcal{X}, \mu). \tag{5.32}$$

Proof. Let F be a function in $\mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda)$. We recall that

$$\nabla F(\xi) = \sum_{i=1}^N \partial_i f(\int h_1 \,\mathrm{d}\xi, \dots, \int h_N \,\mathrm{d}\xi) \int \nabla^E h_i(x) \,\mathrm{d}\xi(x).$$

On the other hand, we can evaluate

$$\begin{aligned}
\mathrm{div}^\mu \nabla F(\xi) &= \sum_{i=1}^N \nabla_{\nabla h_i} \partial_i f(\int h_1 \,\mathrm{d}\xi, \dots, \int h_N \,\mathrm{d}\xi) + \sum_{i=1}^N \partial_i f(\int h_1 \,\mathrm{d}\xi, \dots, \int h_N \,\mathrm{d}\xi) B^c_{\nabla h_i} \\
&= \sum_{i=1}^N \sum_{j=1}^N \partial_i \partial_j f(\int h_1 \,\mathrm{d}\xi, \dots, \int h_N \,\mathrm{d}\xi) \int \langle \nabla^E_x h_i, \nabla^E_x h_j \rangle_E \,\xi(\mathrm{d}x) \\
&\quad \sum_{i=1}^N \partial_i f(\int h_1 \,\mathrm{d}\xi, \dots, \int h_N \,\mathrm{d}\xi) \int \mathrm{div}^\mu \nabla^E h_i(x) \,\xi(\mathrm{d}x) \\
&\quad + \sum_{i=1}^N \partial_i f(\int h_1 \,\mathrm{d}\xi, \dots, \int h_N \,\mathrm{d}\xi) \int \langle \nabla^E \ln c(x, \xi \setminus x), \nabla^E h_i(x) \rangle_E \,\xi(\mathrm{d}x).
\end{aligned}$$

and the identification is complete, once we apply Theorem 5.4.3. Moreover, $\mathcal{H}$ is well-defined as an operator on $\mathrm{L}^2(\mathcal{X}, \mu)$ by Hypothesis 10, as was shown in the proof of Theorem 5.4.4. □

Now, having constructed the generator $\mathcal{H}$ correctly associated to $\mathcal{E}$, Theorem 5.1.1 allows us to conclude to the closability of $\mathcal{E}$. Specifically, we obtain in the same fashion as [2, 3]:

Theorem 5.4.5. *One can define $\mathcal{H}$ which is correctly associated with the pre-Dirichlet form $\mathcal{E}$. As such, the form $(\mathcal{E}, \mathcal{F}C_b^\infty(C^\infty(\Lambda), \mathcal{X}_\Lambda))$ is closable in $\mathrm{L}^2(\mathcal{X}, \mu)$. Its closure is associated with a positive definite self-adjoint operator (Friedrichs extension of $\mathcal{H}$) which is still noted $\mathcal{H}$. It is defined on the extended domain $\overline{\mathrm{Dom}(\mathcal{E})}$.*

Proof. This is a consequence of Theorem 5.1.1. □

Lastly, we also obtain the existence of a quasi-continuous diffusion correctly associated with $\mathcal{E}$, as is given by the following theorem:

Theorem 5.4.6. *Let μ be a point process satisfying the conditions of Theorem 5.4.4. Then there exists a μ-tight special standard process $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$ on $\mathcal{N}$ with transition semigroup*

$$p_t f(\xi) := \mathbb{E}_\xi[f(\mathbf{M}_t)], \quad \xi \in \mathcal{N}, \quad f : \mathcal{N} \longrightarrow \mathbb{R} \quad \textit{square integrable.}$$

In addition, $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$ is properly associated with the Dirichlet form $(\mathcal{E}, \overline{\mathrm{Dom}(\mathcal{E})})$ in the sense that $p_t f$ is an $\mathcal{E}$-a.c., μ-version of $\exp(t\mathcal{H})f$ for all square integrable $f : \mathcal{N} \longrightarrow \mathbb{R}$ and $t > 0$, and such that

$$\mathbf{P}_\xi(\{\omega : \ t \mapsto \mathbf{M}_t(\omega) \textit{ is continuous on } [0, +\infty)\}) = 1, \qquad \mathcal{E}\textit{-a.e., } \xi \in \mathcal{N}, \tag{5.33}$$

i.e. $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$ is quasi-continuous.

Remark 13. *In order to ensure that* $(\mathbf{M}_\mu, (\theta_t)_{t\geq 0})$ *takes its values in* $\mathcal{X}$ *rather than* $\mathcal{N}$*, we have to add the additional hypothesis:*

$$c(x,\eta) \leq c(x,\xi), \quad \forall x \in E,\ \forall \xi \subseteq \eta$$

i.e. the point process has a repulsive nature. In that case, Theorem 5.3.5 still holds, as seen by a careful study of its proof.

Proof. One can apply Theorem 3.4 in [72] (see also the considerations in Section 4 therein) to obtain quasi-regularity of $\mathcal{E}$. Moreover, the proof of locality follows by the same arguments raised in [72]. Therefore, the proof is a direct corollary of the closability of $\mathcal{E}$. □

5.5 Summary

In this chapter, we defined the gradient on the configuration space known as the differential gradient. This gradient was extensively studied in [2, 3] and subsequent papers. We chose as test functions the set of cylindrical functions in order to stay in line with the aforementioned references. The study which was confined to the case of Poisson and Gibbs point processes in [2, 3] is generalized in this chapter to a fully general case, which also includes the determinantal point process which is of specific interest in our manuscript.

The tools developed in this chapter allow us to exhibit the adjoint of the differential gradient ∇ in $\mathrm{L}^2(\mathcal{X}, \mu)$ with respect to a point process μ satisfying only Hypothesis 4. The adjoint generates a diffusion which is shown to be quasi-continuous under an additional hypothesis. These results generalize all known results, and in particular applied to determinantal point processes which was our case at hand. The diffusions obtained in that case were of specific interest since they are known to appear in physics, see e.g. [59, 60].

We strongly suspect the specified integration by parts to hold if and only if the underlying measure in the one with Papangelou intensity c. Unicity of such measure verifying the integration by parts would yield ergodicity of the associated diffusions. Further work in this direction should yield log-Sobolev inequalities in a general setting which is also of further interest.

Chapter 6

Moment formulae

In Section 6.1, we give the main result of the paper, as well as the most interesting consequences. In Section 6.2, we show analogue formulae for $\mathbb{E}[\delta(u)^n]$, where δ is the divergence, which will be rigorously defined in the case of a general point process. To conclude, Section 6.3 will deal with the study of a random transformation of the measure μ.

6.1 Derivation of moment formulae

Let us start by proving a combinatorial lemma.

Lemma 6.1.1. *Let F be a function from the power set $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$. Then for $n \in \mathbb{N}^*$,*

$$\sum_{k=1}^{n+1} \sum_{\mathcal{P} \in \mathcal{T}_{n+1}^k} F(\mathcal{P}) = \sum_{k=1}^{n} \sum_{\mathcal{P}=\{P_1,\dots,P_k\} \in \mathcal{T}_n^k} F\big(\mathcal{P} \cup \big\{\{n+1\}\big\}\big) + \sum_{k=1}^{n} \sum_{\mathcal{P}=\{P_1,\dots,P_k\} \in \mathcal{T}_n^k} \sum_{l=1}^{k} F\big(P_1, \dots, P_{l-1}, P_l \cup \{n+1\}, P_{l+1}, \dots, P_k\big),$$

where $\mathcal{T}_n^k$ is the set of all partitions of $\{1, \dots, n\}$ into k subsets.

Proof. Let us consider the functions $(\Xi_k)_{1 \le k \le n+1}$ defined as follows:

$$\begin{array}{rccl} \Xi_k : & \mathcal{T}_{n+1}^k & \longrightarrow & \mathcal{T}_n^k \cup \mathcal{T}_n^{k-1}, \\ & \{P_1, \dots, P_k\} & \longmapsto & \{P_1, \dots, P_{l-1}, P_l \setminus \{n+1\}, P_{l+1}, \dots, P_k\}, \end{array}$$

where $1 \le l \le k$ is such that $\{n+1\} \in P_l$. Moreover, let us define the function Ξ as

$$\begin{array}{rccl} \Xi : & \mathcal{T}_{n+1} & \longrightarrow & \mathcal{T}_n, \\ & \mathcal{P} & \longmapsto & \Xi_{|\mathcal{P}|}(\mathcal{P}), \end{array}$$

where $\mathcal{T}_n$ is the set of all partitions of $\{1, \dots, n\}$. Then, Ξ is surjective from $\mathcal{T}_{n+1}$ into $\mathcal{T}_n$ and moreover, for $\mathcal{P} = \{P_1, \dots, P_{|\mathcal{P}|}\} \in \mathcal{T}_n$, we have that

$$\Xi^{-1}(\mathcal{P}) = \big\{\mathcal{P} \cup \big\{\{n+1\}\big\}\big\} \cup \bigcup_{l=1}^{|\mathcal{P}|} \big\{P_1, \dots, P_l \cup \{n+1\}, \dots, P_{|\mathcal{P}|}\big\}.$$

Then, using the preceding observations, we obtain

$$\sum_{k=1}^{n+1}\sum_{\mathcal{P}\in\mathcal{T}_{n+1}^k} F(\mathcal{P}) = \sum_{\mathcal{P}\in\mathcal{T}_{n+1}} F(\mathcal{P})$$
$$= \sum_{\substack{\mathcal{P}\in\mathcal{T}_n\\ \mathcal{P}=\{P_1,\dots,P_{|\mathcal{P}|}\}}} F(\mathcal{P}\cup\{n+1\}) + \sum_{l=1}^{|\mathcal{P}|} F(\{P_1,\dots,P_l\cup\{n+1\},\dots,P_{|\mathcal{P}|}\}),$$

which is the desired result once we sum on the different lengths possible for elements of $\mathcal{T}_n$. □

We can now give the first important result of this paper, which will yield many cases of particular interest.

Theorem 6.1.1. *For any $n\in\mathbb{N}$, any measurable nonnegative functions $u_k : E\times\mathcal{X}\to\mathbb{R}$, $k=1,\dots,n$, and any bounded function F on $\mathcal{X}$, we have*

$$\mathbb{E}[F(\xi)\prod_{k=1}^{n}\int u_k(y,\xi)\,\xi(dy)] = \sum_{k=1}^{n}\sum_{\mathcal{P}\in\mathcal{T}_n^k}\mathbb{E}\Big[\int_{E^k} F(\xi\cup x)u^{\mathcal{P}}(x,\xi\cup x)\hat{c}(x,\xi)\,\lambda_k(dx)\Big],$$

where again, $\mathcal{T}_n^k$ denotes the set of all partitions of $\{1,\dots,n\}$ into k subsets. Here, for $\mathcal{P}=\{P_1,\dots,P_k\}\in\mathcal{T}_n^k$, we have used the compact notation $x:=(x_1,\dots,x_k)$, as well as

$$\lambda_k(dx) := \lambda(dx_1)\dots\lambda(dx_k)$$

and

$$u^{\mathcal{P}}(x,\xi) := \prod_{l=1}^{k}\prod_{i\in P_l} u_i(x_l,\xi).$$

Proof. We prove this result by induction on n. Let $u : E\times\mathcal{X}\to\mathbb{R}$ be a measurable nonnegative function. First note that by the Georgii-Nguyen-Zessin identity (3.14), we have

$$\mathbb{E}[F(\xi)\int u(y,\xi)\,\xi(\mathrm{d}y)] = \mathbb{E}[\int F(\xi\cup z)u(z,\xi\cup z)\,c(z,\xi)\,\lambda(\mathrm{d}z)].$$

Now, for $n\in\mathbb{N}^*$, let $u_1,\dots,u_{n+1}$ be nonnegative measurable functions on $E\times\mathcal{X}$. We have by induction:

$$\mathbb{E}[F(\xi)\prod_{k=1}^{n+1}\int u_k(y,\xi)\,\xi(\mathrm{d}y)]$$
$$= \mathbb{E}[F(\xi)\prod_{k=1}^{n}\int u_k(z,\xi)\Big(\int u_{n+1}(y,\xi)\,\xi(\mathrm{d}y)\Big)^{1/n}\xi(\mathrm{d}z)]$$
$$= \sum_{k=1}^{n}\sum_{\mathcal{P}\in\mathcal{T}_n^k}\mathbb{E}\Big[\int_{E^k} F(\xi\cup x)u^{\mathcal{P}}(x,\xi\cup x)\hat{c}(x,\xi)\int u_{n+1}(z,\xi\cup x)\{\xi\cup x\}(\mathrm{d}z)\lambda_k(\mathrm{d}x)\Big].$$

Here, the last part can be rewritten

$$\int u_{n+1}(z,\xi\cup x)\{\xi\cup x\}(\mathrm{d}z) = \int u_{n+1}(z,\xi\cup x)\,\xi(\mathrm{d}z) + \sum_{l=1}^{k} u_{n+1}(x_l,\xi\cup x).$$

Hence, after regrouping the terms, we find

$$\mathbb{E}[F(\xi)\prod_{k=1}^{n+1}\int u_k(y,\xi)\,\xi(\mathrm{d}y)]$$
$$=\sum_{k=1}^{n}\sum_{\mathcal{P}\in\mathcal{T}_n^k}\sum_{l=1}^{k}\mathbb{E}\Big[\int_{E^k}F(\xi\cup x)u^{\mathcal{P}}(x,\xi\cup x)u_{n+1}(x_l,\xi\cup x)\;\hat{c}(x,\xi)\,\lambda_k(\mathrm{d}x)\Big]$$
$$+\sum_{k=1}^{n}\sum_{\mathcal{P}\in\mathcal{T}_n^k}\mathbb{E}\Big[\int_{E^k}F(\xi\cup x)u^{\mathcal{P}}(x,\xi\cup x)\Big(\int u_{n+1}(z,\xi\cup x)\,\xi(\mathrm{d}z)\Big)\,\hat{c}(x,\xi)\,\lambda_k(\mathrm{d}x)\Big].$$

Then, by Fubini's theorem, and the Georgii-Nguyen-Zessin formula (3.14), the expectation in the second sum is equal to

$$\mathbb{E}\Big[\int_{E^k}F(\xi\cup x)u^{\mathcal{P}}(x,\xi\cup x)\Big(\int u_{n+1}(z,\xi\cup x)\,\xi(\mathrm{d}z)\Big)\,\hat{c}(x,\xi)\,\lambda_k(\mathrm{d}x)\Big]$$
$$=\mathbb{E}\Big[\int_{E^{k+1}}F(\xi\cup x\cup z)u^{\mathcal{P}}(x,\xi\cup x\cup z)u_{n+1}(z,\xi\cup x\cup z)\hat{c}(x,\xi\cup z)c(z,\xi)\lambda_k(\mathrm{d}x)\lambda(\mathrm{d}z)\Big]$$
$$=\mathbb{E}\Big[\int_{E^{k+1}}F(\xi\cup x\cup z)u^{\mathcal{P}}(x,\xi\cup x\cup z)u_{n+1}(z,\xi\cup x\cup z)\,\hat{c}(x\cup z,\xi)\,\lambda_k(\mathrm{d}x)\lambda(\mathrm{d}z)\Big]$$
$$=\mathbb{E}\Big[\int_{E^{k+1}}F(\xi\cup x)u^{\mathcal{P}\cup\{n+1\}}(x,\xi\cup x)\,\hat{c}(x,\xi)\,\lambda_{k+1}(\mathrm{d}x)\Big],$$

since by (3.13), we know that $\hat{c}(\{x_1,\ldots,x_k\},\xi\cup y)\,c(y,\xi)=\hat{c}(\{x_1,\ldots,x_k,y\},\xi)$. To summarize, we have found

$$\mathbb{E}[F(\xi)\prod_{k=1}^{n+1}\int u_k(y,\xi)\,\xi(\mathrm{d}y)]$$
$$=\sum_{k=1}^{n}\sum_{\mathcal{P}\in\mathcal{T}_n^k}\Big(\sum_{l=1}^{k}\mathbb{E}\Big[\int_{E^k}F(\xi\cup x)u^{\mathcal{P}}(x,\xi\cup x)u_{n+1}(x_l,\xi\cup x)\;\hat{c}(x,\xi)\,\lambda_k(\mathrm{d}x)\Big]$$
$$+\mathbb{E}\Big[\int_{E^{k+1}}F(\xi\cup x)u^{\mathcal{P}\cup\{n+1\}}(x,\xi\cup x)\,\hat{c}(x,\xi)\,\lambda_{k+1}(\mathrm{d}x)\Big]\Big),$$

and we obtain the desired result by applying lemma 6.1.1. □

The previous quite general property includes many interesting cases. In particular, if all u_k, $k=1,\ldots,n$ are equal, one generalizes the results in [67] and obtains the moments of $\int u_x(\xi)\,\xi(\mathrm{d}x)$:

Corollary 6.1.1. *For any $n\in\mathbb{N}$, any measurable nonnegative function $u:E\times\mathcal{X}\to\mathbb{R}$, and any bounded function F on $\mathcal{X}$, we have*

$$\mathbb{E}[F(\xi)\Big(\int u(y,\xi)\,\xi(dy)\Big)^n]=\sum_{k=1}^{n}\sum_{\{P_1,\ldots,P_k\}\in\mathcal{T}_n^k}\mathbb{E}\Big[\int_{E^k}F(\xi\cup x_1\cup\cdots\cup x_k)$$
$$\prod_{l=1}^{k}u^{|P_l|}(x_l,\xi\cup x_1\cup\cdots\cup x_k)\hat{c}(\{x_1,\ldots,x_k\},\xi)\,\lambda(dx_1)\ldots\lambda(dx_k)\Big],$$

where $|P_i|$ is the cardinality of P_i, $i = 1, \ldots, k$.

This result includes the case where $u(x, \xi) = v(x)$ is a deterministic function:

Corollary 6.1.2. *For any $n \in \mathbb{N}$, and any measurable nonnegative function v on E, we have*

$$\mathbb{E}[F(\xi)\big(\int v(y)\,\xi(dy)\big)^n] = \sum_{k=1}^{n} \sum_{\{P_1,\ldots,P_k\}\in\mathcal{T}_n^k} \int_{E^k} v(x_1)^{|P_1|}\ldots v(x_k)^{|P_k|} \mathbb{E}[F(\xi \cup x_1 \cup \cdots \cup x_k)\hat{c}(\{x_1,\ldots,x_k\},\xi)]\,\lambda(dx_1)\ldots\lambda(dx_k).$$

Corollary 6.1.2 yields

$$\mathrm{Cov}\big(F, (\int v(x)\,\xi(\mathrm{d}x))^n\big) = \sum_{k=1}^{n} \sum_{\{P_1,\ldots,P_k\}\in\mathcal{T}_n^k} \int_{E^k} v(x_1)^{|P_1|}\ldots v(x_k)^{|P_k|} \mathbb{E}[(F(\xi \cup x_1 \cup \cdots \cup x_k) - \mathbb{E}[F(\xi)])\hat{c}(\{x_1,\ldots,x_k\},\xi)]\,\lambda(\mathrm{d}x_1)\ldots\lambda(\mathrm{d}x_k).$$

The case $F = 1$ is also of particular interest:

Corollary 6.1.3. *For any $n \in \mathbb{N}$, any measurable nonnegative non-random functions $v_k : E \to \mathbb{R}$, $k = 1, \ldots, n$, we have*

$$\mathbb{E}[\prod_{k=1}^{n} \int v_k(y)\,\xi(dy)] = \sum_{k=1}^{n} \sum_{\{P_1,\ldots,P_k\}\in\mathcal{T}_n^k} \int_{E^k} \prod_{i\in P_1} v_i(x_1)\ldots\prod_{i\in P_k} v_i(x_k)\,\rho(\{x_1,\ldots,x_k\})\,\lambda(dx_1)\ldots\lambda(dx_k).$$

Proof. Here, it suffices to apply Theorem 6.1.1 to the case of deterministic v_k, $k = 1, \ldots, n$, and use identity (3.16). □

Note that we recover here a classical formula, which reads for $n = 1, 2, 3$:

$$\mathbb{E}[\int v(y)\,\xi(\mathrm{d}y)] = \int_E v(z)\rho(\{z\})\,\lambda(\mathrm{d}z),$$

$$\mathbb{E}[\big(\int v(y)\,\xi(\mathrm{d}y)\big)^2] = \int_E v(z)^2\rho(\{z\})\,\lambda(\mathrm{d}z) + \int_{E^2} v(z)v(y)\rho(\{y,z\})\,\lambda(\mathrm{d}y)\lambda(\mathrm{d}z),$$

$$\mathbb{E}[\big(\int v(y)\,\xi(\mathrm{d}y)\big)^3] = \int_E v(z)^3\rho(\{z\})\,\lambda(\mathrm{d}z) + 3\int_{E^2} v(z)^2v(y)\rho(\{y,z\})\,\lambda(\mathrm{d}y)\lambda(\mathrm{d}z) + \int_{E^3} v(x)v(y)v(z)\rho(\{x,y,z\})\,\lambda(\mathrm{d}x)\lambda(\mathrm{d}y)\lambda(\mathrm{d}z).$$

Moreover, taking $v_k(\cdot) := 1_{A_k}$, where $A_1, \ldots, A_n$ are Borel subsets of E, we obtain the classical formula linking the cumulant measure associated to μ on the left-hand side with the moment measure on the right-hand side (see e.g. [17]). Corollary 6.1.2 is interesting in itself because it is in fact an equivalent characterization of the existence of correlation functions. More precisely, we have the following result:

Proposition 6.1.1. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a family of symmetrical, measurable functions, and $\rho_k : E^k \to \mathbb{R}$ for $k \in \mathbb{N}$. Assume moreover that the measure μ is simple, in the sense that $\mathbb{P}(\xi(x) \leq 1)$ for all $x \in E$. Then, the measure μ possesses correlation functions $(\rho_k)_{k\in\mathbb{N}}$ (with respect to λ) if and only if, for any $n \in \mathbb{N}$, and any measurable nonnegative functions $v_k : E \to \mathbb{R}$, $k = 1, \ldots, n$, we have*

$$\mathbb{E}[\prod_{k=1}^{n} \int v_k(x)\, \xi(dx)] = \sum_{k=1}^{n} \sum_{\{P_1,\ldots,P_k\}\in\mathcal{T}_n^k} \int_{E^k} \prod_{i\in P_1} v_i(x_1) \ldots \prod_{i\in P_k} v_i(x_k) \\ \rho_k(x_1, \ldots, x_k)\, \lambda(dx_1) \ldots \lambda(dx_k).$$

Proof. Assume that we have

$$\int_{\mathcal{X}} \sum_{\alpha\subset\xi,\ \alpha\in\mathcal{X}_0} f(\alpha)\, \mu(\mathrm{d}\xi) = \int_{\mathcal{X}_0} f(\alpha)\, \rho(\alpha)\, L(\mathrm{d}\alpha),$$

where f is any measurable nonnegative function on $\mathcal{X}_0$. Then, for $k \in \mathbb{N}$,

$$\int_{\mathcal{X}} \sum_{\alpha\subset\xi,\ |\alpha|=k} f(\alpha)\, \mu(\mathrm{d}\xi) = \frac{1}{k!} \int_{E^k} f(\{x_1, \ldots, x_k\})\, \rho_k(x_1, \ldots, x_k)\, \lambda(\mathrm{d}x_1) \ldots \lambda(\mathrm{d}x_k).$$

Now, we can write for $n \in \mathbb{N}$ and $\xi \in \mathcal{X}$,

$$\sum_{x_1,\ldots,x_n\in\xi} v_1(x_1) \ldots v_n(x_n) = n! \sum_{k=1}^{n} \sum_{\{P_1,\ldots,P_k\}\in\mathcal{T}_n^k} \sum_{\substack{\alpha\subset\xi,\ |\alpha|=k \\ \alpha=\{x_1,\ldots,x_k\}}} \prod_{i\in P_1} v_i(x_1) \ldots \prod_{i\in P_k} v_i(x_k),$$

where the $n!$ appears since when we write $\{x_1, \ldots, x_k\} \subset \xi$, we only choose ordered subsets of ξ. Then, we find the desired result by taking the expectation of the previous equality. Indeed, we have by definition

$$\mathbb{E}\Big[\sum_{\substack{\alpha\subset\xi,\ |\alpha|=k \\ \alpha=\{x_1,\ldots,x_k\}}} \prod_{i\in P_1} v_i(x_1) \ldots \prod_{i\in P_k} v_i(x_k)\Big] \\ = \frac{1}{n!} \mathbb{E}\Big[\int_{E^k} \prod_{i\in P_1} v_i(x_1) \ldots \prod_{i\in P_k} v_i(x_k) \hat{c}(\{x_1, \ldots, x_k\}, \xi)\, \lambda(\mathrm{d}x_1) \ldots \lambda(\mathrm{d}x_k)\Big],$$

and we conclude by using (3.17).
On the other hand, assume that there exist some symmetrical measurable functions $(\rho_k)_{k\in\mathbb{N}}$, such that for any $n \in \mathbb{N}$, and any measurable nonnegative functions $v^k : E \to \mathbb{R}$, $k = 1, \ldots, n$, we have

$$\mathbb{E}[\prod_{k=1}^{n} \int v_k(y)\, \xi(\mathrm{d}y)] = \sum_{k=1}^{n} \sum_{\{P_1,\ldots,P_k\}\in\mathcal{T}_n^k} \int_{E^k} \prod_{i\in P_1} v_i(x_1) \ldots \prod_{i\in P_k} v_i(x_k) \\ \rho_k(x_1, \ldots, x_k)\, \lambda(\mathrm{d}x_1) \ldots \lambda(\mathrm{d}x_k).$$

Let $A_1, \dots, A_n$ be n disjoint Borel subsets of E. Take $v^k = \mathbf{1}_{A_k}$, $k = 1, \dots, n$. Then, the different terms of the right-hand side sum are all equal to 0, except for the subdivision consisting only of singletons. Hence,

$$\mathbb{E}[\prod_{k=1}^{n} \xi(A_k)] = \int_{A_1 \times \cdots \times A_n} \rho_k(x_1, \dots, x_n)\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n),$$

which means that μ has correlation functions $(\rho_k)_{k \in \mathbb{N}}$ since μ is a simple point process. □

6.2 Extended moment formulae for the divergence

The aim of this section is to obtain analogue formulae involving the divergence of a general operator. Let us now start by defining a new signed random measure ν by the formula

$$\int f(y, \xi)\, \nu(\mathrm{d}y) = \int f(y, \xi)\, \xi(\mathrm{d}y) - \int f(y, \xi) c(y, \xi)\, \lambda(\mathrm{d}y),$$

for f on $E \times \mathcal{X}$ such that $\mathbb{E}[\int_E |f(y, \xi)| c(y, \xi)\, \lambda(\mathrm{d}y)] < \infty$. We remark that the (signed) measure ν is not defined on the whole Borel σ-algebra, and hence we need to make another assumption on μ which is assumed to hold henceforth.

Hypothesis 11. *We assume that* $\forall i \in \mathbb{N}$, $\forall k \in \mathbb{N}^*$,

$$\mathbb{E}[\xi(\Lambda)^k \big(\int_\Lambda c(z, \xi)\, \lambda(dz) \big)^i] < \infty,$$

for all compact sets $\Lambda \subset E$.

We can remark that the previous condition implies that the moments of the point process μ are all finite as defined in (5.7), i.e. that for all φ bounded, compactly supported function, and for all $n \in \mathbb{N}$,

$$\mathbb{E}[\big(\int \varphi(y)\, \xi(\mathrm{d}y) \big)^n] < \infty. \tag{6.1}$$

We now give some adequate conditions implying the previous hypothesis.

Proposition 6.2.1. *Assume (6.1), i.e. that the point process μ admits finite moments. Moreover, assume that one of the following conditions is verified.*

(i) There exist a locally integrable function f on E and a bounded function F on $\mathcal{X}$ such that $\forall x \in E, \forall \xi \in \mathcal{X}, c(x, \xi) \le f(x) F(\xi)$.

(ii) The point process μ is repulsive, in the sense that $\forall \xi, \eta \in \mathcal{X}$, $\xi \subset \eta$, we have $c(x, \xi) \ge c(x, \eta)$.

(iii) The point process μ is attractive, in the sense that $\forall \xi, \eta \in \mathcal{X}$, $\xi \subset \eta$, we have $c(x, \xi) \le c(x, \eta)$.

Then, Hypothesis 11 is satisfied.

Proof. The case (i) is trivial. Assume that (ii) is verified. Then, for all $x \in \Lambda$, $\xi \in \mathcal{X}$, $c(x, \xi) \le c(x, \emptyset) = \frac{j_\Lambda(x)}{j_\Lambda(\emptyset)}$, where j_Λ is the Janossy density of μ. Then, we have for all f on

E,

$$\begin{aligned}\int_\Lambda f(z)c(z,\emptyset)\,\lambda(\mathrm{d}z) &= \mathbb{E}[\int_\Lambda f(z)c(z,\xi)1_{\{\xi=\emptyset\}}\,\lambda(\mathrm{d}z)]\\ &= \mathbb{E}[\int_\Lambda f(z)1_{\{\xi\setminus z=\emptyset\}}\,\xi(\mathrm{d}z)]\\ &= \mathbb{E}[f(\xi)1_{|\xi_\Lambda|=1}]\\ &= \int f(z)j_\Lambda(z)\,\lambda(\mathrm{d}z).\end{aligned}$$

Hence,

$$\mathbb{E}[\xi(\Lambda)^k\big(\int_\Lambda c(z,\xi)\,\lambda(\mathrm{d}z)\big)^i] \le \frac{1}{j_\Lambda(\emptyset)^i}\mathbb{E}[\xi(\Lambda)^k](\int_\Lambda j_\Lambda(z)\,\lambda(\mathrm{d}z))^i < \infty.$$

To conclude the proof, assume that (iii) is verified. Then, $\forall x,y\in\Lambda$, $\xi\in\mathcal{X}$, we have $c(x,\xi)\le c(x,\xi\cup y)$. Hence,

$$\prod_{k=1}^i c(x_k,\xi) \le c(x_1,\xi)c(x_2,\xi\cup x_1)\dots c(x_i,\xi\cup x_1\cup\dots\cup x_{i-1}) = \hat{c}(\{x_1,\dots,x_i\},\xi).$$

Therefore, by applying (3.15) to $u(\alpha,\xi)=1_{\{|\alpha|=i\}}\xi(\Lambda)^k$, we obtain

$$\mathbb{E}[\xi(\Lambda)^k\big(\int_\Lambda c(z,\xi)\,\lambda(\mathrm{d}z)\big)^i] \le \mathbb{E}\big[\max(\xi(\Lambda)-i,0)^k\frac{\xi(\Lambda)!}{(\xi(\Lambda)-i)!}\big] < \infty.$$

□

Then, we prove the following:

Proposition 6.2.2. *We assume that Hypothesis 11 is verified. Then, for any $n\in\mathbb{N}$, any bounded process $u: E\times\mathcal{X}\to\mathbb{R}$ with compact support on E, and any bounded function F on $\mathcal{X}$, we have*

$$\mathbb{E}[F(\xi)\big(\int u(y,\xi)\,\nu(dy)\big)^n] = \sum_{i=0}^n(-1)^i\binom{n}{i}\sum_{k=1}^{n-i}\sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_{n-i}^k}$$

$$\mathbb{E}\Big[\int_{E^k}\hat{c}(x,\xi)F(\xi\cup x)\prod_{l=1}^k u(x_l,\xi\cup x)^{|P_l|}\big(\int u(z,\xi\cup x)c(z,\xi\cup x)\,\lambda(dz)\big)^i\lambda_k(dx)\Big]. \tag{6.2}$$

Proof. This is a direct consequence of the binomial formula

$$\begin{aligned}&\mathbb{E}[F(\xi)\big(\int u(y,\xi)\,\nu(\mathrm{d}y)\big)^n]\\ &\quad=\sum_{i=0}^n(-1)^i\binom{n}{i}\mathbb{E}\Big[F(\xi)\big(\int u(y,\xi)\,\xi(\mathrm{d}y)\big)^{n-i}\big(\int u(y,\xi)c(y,\xi)\,\lambda(\mathrm{d}y)\big)^i\Big],\end{aligned}$$

and thus we obtain the desired result by applying Theorem 6.1.1. It remains to show that the right-hand side of (6.2) is well-defined. Taking into account the conditions on u and F, each of the terms in the (finite) sum is smaller in absolute value than

$$\begin{aligned}C\,\mathbb{E}[\int_{\Lambda^k}\hat{c}(x,\xi)\big(\int_\Lambda c(z,\xi\cup x)\,\lambda(\mathrm{d}z)\big)^i\lambda_k(\mathrm{d}x)] = C\,\mathbb{E}[\frac{|\xi_\Lambda|!}{(|\xi_\Lambda|-k)!}\big(\int_\Lambda c(z,\xi\cup x)\,\lambda(\mathrm{d}z)\big)^i]\\ <\infty,\end{aligned}$$

where Λ is the support of u, C is a nonnegative constant, and we have used (3.15) with $u(\alpha,\xi)=1_{\{|\alpha|=k\}}\big(\int c(z,\xi\cup\alpha)\,\lambda(\mathrm{d}z)\big)^i$. □

Recall that the associated formula for π^λ (the Poisson process of intensity λ), as obtained in [67], is

$$\mathbb{E}_{\pi^\lambda}[F(\xi)\big(\int u(y,\xi)\,\nu(\mathrm{d}y)\big)^n]=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{k=1}^{n-i}\sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_{n-i}^k}\mathbb{E}_{\pi^\lambda}\Big[\int_{E^{i+k}}F(\xi\cup\{x_1\cup\cdots\cup x_k\})$$
$$\prod_{l=1}^{k}u^{|P_l|}(x_l,\xi\cup x_1\cup\cdots\cup x_k)\prod_{l=k+1}^{i+k}u(x_l,\xi\cup x_1\cup\cdots\cup x_k)\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_{i+k})\Big].$$

Here, by comparison, the general formula can be written as

$$\mathbb{E}[F(\xi)\big(\int u(y,\xi)\,\nu(\mathrm{d}y)\big)^n]$$
$$=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{k=1}^{n-i}\sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_{n-i}^k}\mathbb{E}\Big[\int_{E^{i+k}}F(\xi\cup\{x_1\cup\cdots\cup x_k\})G_k(\{x_1,\dots,x_{i+k}\},\xi)$$
$$\prod_{l=1}^{k}u^{|P_l|}(x_l,\xi\cup x_1\cup\cdots\cup x_k)\prod_{l=k+1}^{i+k}u(x_l,\xi\cup x_1\cup\cdots\cup x_k)\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_{i+k})\Big],$$

where

$$G_k(\{x_1,\dots,x_{i+k}\},\xi)$$
$$=\hat{c}(\{x_1,\dots,x_k\},\xi)c(x_{i+1},\xi\cup x_1\cup\cdots\cup x_k)\dots c(x_{i+k},\xi\cup x_1\cup\cdots\cup x_k)$$
$$=\prod_{l=1}^{k}c(x_l,\xi\cup x_1\cup\cdots\cup x_{l-1})\prod_{l=i+1}^{i+k}c(x_l,\xi\cup x_1\cup\cdots\cup x_k).$$

Let us now introduce the so-called divergence operator δ.

Definition 15 (Divergence operator)**.** *For any measurable $u:E\times\mathcal{X}\to\mathbb{R}$ such that $\mathbb{E}[\int|u(y,\xi)|\,c(y,\xi)\,\lambda(dy)]<\infty$, we define $\delta(u)$ as*

$$\delta(u)=\int u(y,\xi\setminus y)\,\nu(dy)=\int u(y,\xi\setminus y)\,\xi(dy)-\int u(y,\xi)\,c(y,\xi)\,\lambda(dy).\tag{6.3}$$

Note that what we call divergence in this paper is also called residual process, or innovation process, see e.g. [18]. We can notice that the divergence of a bounded process u, which has support $\Lambda\times\mathcal{X}$, is correctly defined since for such a process,

$$\mathbb{E}[\int|u(y,\xi)|\,c(y,\xi)\,\lambda(dy)]\le M\,\mathbb{E}[\int_\Lambda c(y,\xi)\,\lambda(dy)]=M\int_\Lambda\rho_1(x)\,\lambda(\mathrm{d}x)<\infty,$$

where for almost every $(x,\xi)\in\Lambda\times\mathcal{X}$, $u(x,\xi)\le M$.
The next proposition gives a moment formula for this newly introduced operator.

Proposition 6.2.3. *For any $n \in \mathbb{N}$, any bounded process $u : E \times \mathcal{X} \to \mathbb{R}$, with compact support on E, and any bounded function F on $\mathcal{X}$, we have*

$$\mathbb{E}[F(\xi)\big(\delta(u)\big)^n] = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{k=1}^{n-i}\sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_{n-i}^k}\mathbb{E}\Big[\int_{E^{i+k}}F(\xi\cup\{x_1,\dots,x_k\})G_k(\{x_1,\dots,x_{i+k}\},\xi)$$
$$\prod_{l=k+1}^{i+k}u(x_l,\xi\cup\{x_1,\dots,x_k\})\prod_{l=1}^{k}u^{|P_l|}(x_l,\xi\cup\{x_1,\dots,\tilde{x_l},\dots,x_k\})\lambda(dx_1)\dots\lambda(dx_{i+k})\Big],$$

where, G_k is the function defined in the previous paragraph. The notation $\{x_1,\dots,\tilde{x_l},\dots,x_k\}$ stands for $\{x_1,\dots,x_{l-1},x_{l+1},\dots,x_k\}$.

Proof. Recall that by definition,

$$\mathbb{E}[F(\xi)\big(\delta(u)\big)^n] = \mathbb{E}[F(\xi)\big(\int v(y,\xi)\,\nu(\mathrm{d}y)\big)^n],$$

where $v(y,\xi) = u(y,\xi\setminus y)$, for $(y,\xi)\in E\times\mathcal{X}$. Therefore, we can apply Proposition 6.2.2, and we obtain the desired result. □

The previous definition of the operator δ is justified by the duality formula with respect to the difference gradient, defined below.

Definition 16 (Difference operator)**.** *For $F : \mathcal{X}\to\mathbb{R}$, we define DF, the difference operator applied to F, as follows:*

$$\begin{array}{rccc} DF: & E\times\mathcal{X} & \longrightarrow & \mathbb{R} \\ & (x,\xi) & \longmapsto & D_xF(\xi) = F(\xi\cup x) - F(\xi\setminus x). \end{array}$$

Remark 14. *The difference operator is equal to $D_xF(\xi) = F(\xi\cup x) - F(\xi)$, $(\lambda\otimes\mu)$-a.s.. However, we wish to study DF as a stochastic process under the measure C_μ and therefore the previous definition is the relevant one.*

Then, the previous definition of δ is an operator satisfying the following corollary of Proposition 6.2.3.

Corollary 6.2.1 (Duality relation)**.** *For any bounded function F on $\mathcal{X}$, and a process $u : (x,\xi)\mapsto u(x,\xi)$, C_μ-integrable, and in* Dom(δ)*, we have*

$$\mathbb{E}[F(\xi)\delta(u)] = \mathbb{E}[\int_E D_zF(\xi)u(z,\xi)c(z,\xi)\,\lambda(dz)]. \tag{6.4}$$

Here, we say that a process $u : (x,\xi)\mapsto u(x,\xi)$ belongs to Dom(δ) whenever there exists a constant c such that for any $F\in \mathrm{L}^2(\mathcal{X},\mu)$,

$$\Big|\int_{\mathcal{X}}\int_E D_zF(\xi)u(z,\xi)\,C_\mu(\mathrm{d}z,\mathrm{d}\xi)\Big| \le c\|F\|_{\mathrm{L}^2(\mathcal{X},\mu)}.$$

Recall that in the Poisson case, $C_\mu = \lambda\otimes\mu$ and we recover the classical duality relationship. One also finds an extended Skohorod isometry in the case of a more general point process (see [18]):

Corollary 6.2.2. *For a process $u : (x,\xi) \mapsto u(x,\xi)$, C_μ-integrable, and in* $\mathrm{Dom}(\delta)$*, we have*

$$\begin{aligned}\mathbb{E}[\delta(u)^2] = \mathbb{E}[\int u(y,\xi)^2 c(y,\xi)\,\lambda(dy)] \\ + \mathbb{E}[\int D_z u(y,\xi) D_y u(z,\xi) \hat{c}(\{x,y\},\xi)\,\lambda(dy)\lambda(dz)] \\ - \mathbb{E}[\int\int u(z,\xi)u(y,\xi)c(z,\xi)D_z c(y,\xi)\,\lambda(dy)\lambda(dz)].\end{aligned}$$

This particular way of writing the formula is useful since if we assume that u verifies $D_z u(y,\xi)D_y u(z,\xi) = 0$ for a.e. $x, y \in E$ and $\xi \in \mathcal{X}$, then the second term is zero. This assumption is equivalent to adaptedness when $E = \mathbb{R}$, see [67] for example. Moreover, one would have the expected Skorohod isometry $\mathbb{E}[\delta(u)^2] = \mathbb{E}[\int u(y,\xi)^2 c(y,\xi)\,\lambda(dy)]$ if and only if $c(y,\xi) = c(y,\xi \cup z)$ for a.e. $(y,z) \in E^2$ and a.e. $\xi \in \mathcal{X}$, i.e. if c does not depend on the configuration ξ. Hence, μ would therefore necessarily be a PPP in this case (see [53]).

Let us conclude this section by giving an example of point processes satisfying Hypothesis 11.
We return to the case of determinantal point processes defined previously. We further define the operator J, called the global interaction operator, as $J := K(I-K)^{-1}$. Since we assumed that $||K|| < 1$, J is properly defined as a bounded operator. We also define $J_{[\Lambda]} := K_\Lambda (I - K_\Lambda)^{-1}$, where we are wary of the fact that $J_{[\Lambda]}$ is not the restriction of J to Λ, but rather J applied to the operator K_Λ. Then, it was proved in [28] that $\mu_{K,\Lambda}$ (restriction of μ to Λ) admits Papangelou intensities given by $c_{K,\Lambda}(x,\xi) = \frac{\det J_{[\Lambda]}(\xi \cup x)}{\det J_{[\Lambda]}(\xi)}$, $x \in E, \xi \in \mathcal{X}$. Here, the notation $J_{[\Lambda]}(\xi)$ stands for $\big(J_{[\Lambda]}(x_i,x_j)\big)_{1\le i,j\le n}$ where $\xi = \{x_1,\dots,x_n\} \in \mathcal{X}_\Lambda$. However, the main result of [28] is as follows:

Theorem 6.2.1 (c.f. Theorem 3.6 of [28])**.** *(μ_K, λ) also satisfies condition (Σ_λ) and its Papangelou intensity c_K is given by*

$$c_K(x,\xi) = \lim_{n\to\infty} c_{K,\Lambda_n}(x,\xi_{\Lambda_n}),$$

where $(\Lambda_n)_{n\in\mathbb{N}}$ is a sequence of compact sets of E that increases to E. Moreover, (μ_K,λ) is repulsive in the sense that

$$c(x,\xi) \ge c(x,\eta),$$

for almost every $x \in E$ and $\xi, \eta \in \mathcal{X}$, $\xi \subset \eta$.

Hence, taking into consideration the previous results, (μ_K,λ) satisfies Proposition 6.2.1 (ii) and therefore, (μ_K,λ) satisfies Hypothesis 11.

6.3 Random transformation of the point process

The goal of this section is to give an application of the previous moment formulae. As was explained in the introduction, we wish to study a random transformation of the point process measure μ. Hereafter, we consider the following condition.

Hypothesis 12. *For $u : E \times \mathcal{X} \to \mathbb{R}$, we assume that u satisfies*

$$D_{x_1} \dots D_{x_k} u(x_1,\xi) \dots u(x_k,\xi) = 0, \qquad \text{for } C_\mu\text{-a.e. } (x,\xi) \in E \times \mathcal{X}$$

for all $k \ge 1$.

Before the proof of the main result, we give a few mostly combinatorial results that will be useful in the following. To this end, let us define, for $x_1, \dots, x_n \in E$ and $\Theta = \{1, \dots, k\}$, $k \in \mathbb{N}^*$,

$$D_\Theta = D_{x_1} \circ \cdots \circ D_{x_k},$$

where D is the difference operator. Moreover, recall that D satisfies the following property:

$$D(FG) = GDF + FDG + (DF)(DG), \tag{6.5}$$

for any random variables $F, G : \mathcal{X} \to \mathbb{R}$. Additionally, the previous equation can be generalized to the case of $F_1, \dots, F_n : \mathcal{X} \to \mathbb{R}$ in the following way:

$$D_\Theta F_1 \dots F_n = \sum_{\substack{I_1, \dots, I_n \subseteq \Theta \\ I_1 \cup \cdots \cup I_n = \Theta}} = (D_{I_1} F_1) \dots (D_{I_n} F_n),$$

where $n \in \mathbb{N}^*$ and $\Theta \subset \{1, \dots, n\}$. Then, we prove the following factorization property of the difference operator:

Lemma 6.3.1. *For any $n \in \mathbb{N}^*$, we take $\Theta \subseteq \{1, \dots, n\}$. Then, for any $u : E \times \mathcal{X} \to \mathbb{R}$, assume that Hypothesis 12 is verified. Then, we have that*

$$D_\Theta u(x_1, \xi) \dots u(x_n, \xi) = \sum_{M \subseteq \Theta \ ; \ M \neq \emptyset} (-1)^{1+|M|} \prod_{i \in I} u(x_i, \xi)\, D_\Theta \prod_{i \in \{1, \dots, n\} \setminus M} u(x_i, \xi),$$

for C_μ-a.e. $(x, \xi) \in E \times \mathcal{X}$.

Proof. The proof of this lemma is mostly combinatorial. Take $\Theta \subseteq \{1, \dots, n\}$, then for C_μ-a.e. $(x, \xi) \in E \times \mathcal{X}$, we have

$$\begin{aligned}
&D_\Theta u(x_1, \xi) \dots u(x_n, \xi) \\
&\quad = \sum_{\substack{I_1, \dots, I_{|\Theta|}, I_{|\Theta|+1} \\ I_1 \cup \cdots \cup I_{|\Theta|+1} = \Theta}} \left(\prod_{k \in \Theta} D_{I_k} u(x_k, \xi) \right) D_{I_{|\Theta|+1}} \prod_{i \in \{1, \dots, n\} \setminus \Theta} u(x_i, \xi) \\
&\quad = \sum_{\substack{K \subseteq \Theta \\ K \neq \emptyset}} \sum_{\substack{I_1, \dots, I_{|\Theta|-|K|}, I_{|\Theta|-|K|+1} \subseteq \Theta \\ I_1 \neq \emptyset, \dots, I_{|\Theta|-|K|+1} \neq \emptyset, I_1 \cup \cdots \cup I_{|\Theta|-|K|+1} = \Theta}} \prod_{i \in K} u(x_i, \xi) \\
&\qquad\qquad \left(\prod_{i \in \Theta \setminus K} D_{I_i} u(x_i, \xi) \right) D_{I_{|\Theta|-|K|+1}} \prod_{i \in \{1, \dots, n\} \setminus \Theta} u(x_i, \xi) \\
&\quad = \sum_{\substack{K \subseteq \Theta \\ K \neq \emptyset}} \sum_{L \subseteq \Theta \setminus K} (-1)^{|L|} \prod_{i \in K \cup L} u(x_i, \xi)\, D_\Theta \prod_{i \in \{1, \dots, n\} \setminus K \setminus L} u(x_i, \xi) \\
&\quad = \sum_{\substack{M \subseteq \Theta \\ M \neq \emptyset}} \prod_{i \in M} u(x_i, \xi)\, D_\Theta \prod_{i \in \{1, \dots, n\} \setminus M} u(x_i, \xi) \sum_{J \subseteq M} (-1)^{|J|} \\
&\quad = \sum_{\substack{M \subseteq \Theta \\ M \neq \emptyset}} (-1)^{|M|+1} \prod_{i \in M} u(x_i, \xi)\, D_\Theta \prod_{i \in \{1, \dots, n\} \setminus M} u(x_i, \xi),
\end{aligned}$$

where in the previous equalities, $K, M \neq \emptyset$ because of Hypothesis 12. □

Lemma 6.3.2. *Let δ be the divergence operator defined in (6.3). Then, for any integrable $u : E \times \mathcal{X} \to \mathbb{R}$, we have that*

$$\delta\left(\hat{c}(\{x_1,\dots,x_n\},\xi\cup\cdot)u(\cdot,\xi)\right) = \hat{c}(\{x_1,\dots,x_n\},\xi)\left(\delta(u)+\int u(z,\xi)(c(z,\xi)-c(z,\xi\cup\{x_1,\dots,x_n\}))\,\lambda(dz)\right),$$

for $x_1,\dots,x_n \in E$ and $\xi \in \mathcal{X}$.

Proof. The proof follows from the definition (6.3), and the fact that

$$c(z,\xi)\hat{c}(\{x_1,\dots,x_n\},\xi\cup z)c(z,\xi) = \hat{c}(\{x_1,\dots,x_n,z\},\xi) = \hat{c}(\{x_1,\dots,x_n\},\xi)c(z,\xi\cup\{x_1,\dots,x_n\}),$$

which follows from (3.13). □

Now, let us consider a random shifting $\tau : E \times \mathcal{X} \to E$. For $\xi \in \mathcal{X}$, consider the image measure of ξ by τ, denoted by $\tau_*(\xi)$, and defined as

$$\tau_*(\xi) = \sum_{x\in\xi} \delta_{\tau(x,\xi)},$$

and thus τ_* shifts each point of the configuration in the direction τ. Now, we wish to study the effect of the the transformation on the underlying measure μ under sufficiently strong conditions on τ. The following hypotheses will be considered:

(HT1) The random transformation τ_* satisfies Hypothesis 12, in the sense that for any $u : \mathcal{X} \to \mathbb{R}$, $u \circ \tau_*$ verifies Hypothesis 12.

(HT2) For a.e. $\xi \in \mathcal{X}$, $\tau(\cdot,\xi)$ is invertible, and we denote its inverse by $\tau^{-1}(x,\xi)$, $x \in E, \xi \in \mathcal{X}$. We also denote by $\tau_*^{-1}(\xi)$ the image measure of ξ by τ^{-1}.

We are now in a position to prove the main theorem of this section.

Theorem 6.3.1. *Let $\tau : E \times \mathcal{X} \to E$ be a random shifting as defined previously, and satisfying* **(HT1)** *and* **(HT2)**. *Let us assume that τ maps λ to σ, i.e. $\tau(\cdot,\xi)\lambda = \sigma$, $\xi \in \mathcal{X}$, where σ is a fixed measure on $(E,\mathcal{B})$. Then, $\tau_*\mu$ has correlation functions with respect to σ that are given by*

$$\rho_\tau(x_1,\dots,x_k) = \mathbb{E}\big[\hat{c}(\{\tau^{-1}(x_1,\xi),\dots,\tau^{-1}(x_k,\xi)\},\xi)\big], \quad x_1,\dots,x_k \in E. \tag{6.6}$$

Proof. Our aim for the proof is to use Proposition 6.1.1 (characterization of correlation functions). By Theorem 6.1.1, we have for any $n \in \mathbb{N}$, any measurable nonnegative (non-random) functions $v_k : E \to \mathbb{R}$, $k = 1,\dots,n$,

$$\mathbb{E}_{\tau_*\mu}\Big[\prod_{k=1}^n \int v_k(y)\,\xi(\mathrm{d}y)\Big] = \sum_{k=1}^n \sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_n^k} \mathbb{E}_\mu\Big[\int_{E^k} \hat{c}(\{x_1,\dots,x_k\},\xi)$$

$$\prod_{l=1}^{k}\prod_{i\in P_l} v_i(\tau(x_l,\xi\cup x_1\cup\cdots\cup x_k))\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_k)\Big]$$

$$=\sum_{k=1}^{n}\sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_n^k}\mathbb{E}_\mu\Big[\int_{E^k}\hat{c}(\{x_1,\dots,x_k\},\xi)$$

$$\sum_{\Theta\subset\{1,\dots,k\}} D_\Theta\prod_{l=1}^{k}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_k)\Big]$$

$$=\sum_{k=1}^{n}\sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_n^k}\sum_{\Theta\subseteq\{1,\dots,k\}}\sum_{M\subseteq\Theta,\ M\neq\emptyset}(-1)^{|M|+1}\mathbb{E}_\mu\Big[\int_{E^k}\hat{c}(\{x_1,\dots,x_k\},\xi)$$

$$\prod_{l\in M}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,D_\Theta\prod_{l\notin M}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_k)\Big]$$

where the first equality follows from Theorem 6.1.1, the second equality follows from

$$\prod_{i=1}^{k}(\mathrm{I}+D_{x_k})=\sum_{\Theta\subset\{1,\dots,k\}} D_\Theta$$

as an operator equality, and the third equality follows from Lemma 6.3.1. Then, we proceed by doing $|\Theta|$ integration by parts according to (6.4) in order to be able to do a change of variable in the previous equation. Indeed, take $k\in\{1,\dots,n\}$, $\{P_1,\dots,P_k\}\in\mathcal{T}_n^k$, and $\Theta\subset\{1,\dots,k\}$, $\Theta\neq\emptyset$. Assume for example that $\Theta=\{1,\dots,p\}$, where $p:=|\Theta|\leq k$. Then,

$$\mathbb{E}_\mu\Big[\int_{E^k}\hat{c}(\{x_1,\dots,x_k\},\xi)$$

$$\prod_{l\in\Theta}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,D_\Theta\prod_{l\notin\Theta}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_k)\Big]$$

$$=\mathbb{E}_\mu\Big[\int_{E^k} c(x_1,\xi)\hat{c}(\{x_2,\dots,x_k\},\xi\cup x_1)$$

$$\prod_{l\in\Theta}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,D_\Theta\prod_{l\notin\Theta}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_k)\Big]$$

$$=\mathbb{E}_\mu\Big[\int_{E^k}\delta\Big(\hat{c}(\{x_2,\dots,x_k\},\xi\cup\cdot)\prod_{i\in P_1} v_i(\tau(\cdot,\xi))\prod_{l\in\Theta\setminus\{1\}}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\Big)$$

$$D_{\Theta\setminus\{1\}}\prod_{l\notin\Theta}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,\lambda(\mathrm{d}x_2)\dots\lambda(\mathrm{d}x_k)\Big]$$

$$=\mathbb{E}_\mu\Big[\int_{E^k}\hat{c}(\{x_2,\dots,x_k\},\xi)\Big(\delta(\prod_{i\in P_1} v_i(\tau(\cdot,\xi))\prod_{l\in\Theta\setminus\{1\}}\prod_{i\in P_l} v_i(\tau(x_l,\xi)))$$

$$+\int\prod_{i\in P_1} v_i(\tau(z,\xi))\prod_{l\in\Theta\setminus\{1\}}\prod_{i\in P_l} v_i(\tau(x_l,\xi))(c(z,\xi)-c(z,\xi\cup\{x_1,\dots,x_n\})\,\lambda(\mathrm{d}z)\Big)$$

$$D_{\Theta\setminus\{1\}}\prod_{l\notin\Theta}\prod_{i\in P_l} v_i(\tau(x_l,\xi))\,\lambda(\mathrm{d}x_2)\dots\lambda(\mathrm{d}x_k)\Big],$$

where the last line results from Lemma 6.3.2. By iterating these integration by parts p times, and recombining the different parts, we obtain

$$\mathbb{E}_{\tau_*\mu}[\prod_{k=1}^{n}\int v_k(y)\,\xi(\mathrm{d}y)] = \sum_{k=1}^{n}\sum_{\{P_1,\dots,P_k\}\in\mathcal{T}_n^k}\mathbb{E}_{\mu}\Big[\int_{E^k}\hat{c}(\{x_1,\dots,x_k\},\xi)\prod_{l=1}^{k}\prod_{i\in P_l}v_i(\tau(x_l,\xi))\,\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_k)\Big],$$

and it suffices to do a change of variables on E. Then, we conclude by Proposition 6.1.1, that $\tau_*\mu$ has correlation functions (with respect to σ), which are given by

$$\rho_\tau(x_1,\dots,x_k) = \mathbb{E}\big[\hat{c}(\{\tau^{-1}(x_1,\xi),\dots,\tau^{-1}(x_k,\xi)\},\xi)\big],\quad x_1,\dots,x_k\in E,$$

since the previous functions are obviously symmetrical (by symmetry of $\hat{c}(\cdot,\xi)$ as a function on E^n), and since an invertible transformation of a simple point process μ yields another simple process (allowing us to apply Proposition 6.1.1). □

This theorem directly generalizes all known results. Let us start by considering the following corollary, which was obtained in [67].

Corollary 6.3.1. *Let $\mu=\pi^{d\lambda}$ be the Poisson measure with intensity λ. Let $\tau: E\times\mathcal{X}\to E$ be a random transformation satisfying* (**HT1**) *and* (**HT2**). *Let us assume that τ maps λ to σ, i.e. $\tau(\cdot,\xi)\lambda=\sigma$, $\xi\in\mathcal{X}$. Then, τ maps $\pi^{d\lambda}$ to $\pi^{d\sigma}$.*

Proof. The corollary follows directly from the theorem, since $\pi^{\mathrm{d}\lambda}$ has a Papangelou intensity of 1. Therefore, $\tau_*\pi^{\mathrm{d}\lambda}$ has intensity σ and its correlation functions, given by (6.6), are also equal to 1. □

We can also apply these results to the case of the determinantal point process. In order to ensure a closed form of the Papangelou intensities, we recall that Hypothesis 5 needs to hold. More precisely, it was shown that under Hypothesis 5, the determinantal process with global interaction operator J satisfies condition (Σ_λ), i.e. it admits Papangelou intensities c_K. Moreover, under condition Hypothesis 5, c_K is explicitly given by (3.23) which we recall here:

$$c_K(x,\xi) = \frac{\det J(\xi_W\cup x)}{\det J(\xi_W)}\mathbf{1}_{\mathrm{diam}\ W(x,\xi)<\infty},$$

where $W(x,\xi)$ is the union of the clusters of $B_R(x\cup\xi)$ hitting x and $\xi_W := \xi_{W(x,\xi)}$. Therefore, we obtain the following corollary of Theorem 6.3.1:

Corollary 6.3.2. *Let μ_K be the determinantal measure with intensity λ and kernel K. Assume that the associated operator J satisfies Hypothesis 5. Let $\tau: E\times\mathcal{X}\to E$ be a transformation satisfying* (**HT1**) *and* (**HT2**). *Let us assume that τ maps λ to σ, i.e. $\tau(\cdot,\xi)\lambda=\sigma$, $\xi\in\mathcal{X}$. Then, $\tau_*\mu_K$ has correlation functions given by*

$$\rho_\tau(x_1,\dots,x_k) = \mathbb{E}\big[\frac{\det J(\xi_W\cup\{\tau^{-1}(x_1,\xi),\dots,\tau^{-1}(x_k,\xi)\})}{\det J(\xi_W)}\big],\quad x_1,\dots,x_k\in E, \tag{6.7}$$

and where W is the union of the clusters of $B_R(\{\tau^{-1}(x_1,\xi),\dots,\tau^{-1}(x_k,\xi)\}\cup\xi)$ hitting $\{\tau^{-1}(x_1,\xi),\dots,\tau^{-1}(x_k,\xi)\}$. Moreover, if we further assume that τ is a non-random, invertible transformation, and define

$$\begin{aligned} T:\ & \mathrm{L}^2(E,\sigma)\longrightarrow \mathrm{L}^2(E,\lambda),\\ & f\ \longmapsto f\circ\tau. \end{aligned}$$

Then, τ maps (μ_K,λ) to (μ_{K_τ},σ), where $K_\tau = T^{-1}KT$.

Proof. The first part of the corollary is a direct consequence of Theorem 6.3.1. If we further assume that τ is non-random, then by (3.17), we have

$$\rho_\tau(x_1,\dots,x_k)=\rho(\tau^{-1}(x_1),\dots,\tau^{-1}(x_n))=\det\left(K(\tau^{-1}(x_i),\tau^{-1}(x_j))\right)_{1\le i,j\le n}.$$

It then remains to notice that the kernel $K_\tau = T^{-1}KT$ is again an integral operator with kernel $K_\tau(x,y)=K(\tau^{-1}(x),\tau^{-1}(y))$, $x,y\in E$. □

Remark 15. *If we do not assume that τ is non-random, then there is no reason for ρ_τ to be given by the determinant of a Hilbert-Schmidt operator K'. Therefore, $\tau_*\mu_K$ is not necessarily determinantal in the general case of a random shift τ.*

We can note that the last part of the corollary is another formulation of the quasi-invariance results obtained in [14]. The study in the aforementioned paper was limited to determinantal processes on a compact set Λ. On such a compact set, (**H3**) is obviously satisfied and we therefore have the existence of Papangelou intensities in the form given above.

6.4 Summary

In this chapter, we defined a stochastic integral on the configuration space. We consider the most general case of an integrand depending not only on the underlying space, but also on the whole configuration $\xi\in\mathcal{X}$. This general case is to be compared with the anticipative case which is studied extensively in the brownian motion case.

Here, we proceed along the same lines as in [67] in that we obtain moment formulae for any moment of the stochastic integral. However, on the contrary of [67] which restricts itself to the Poisson point process, we obtain general results valid for any point process satisfying Hypothesis 4. We emphasize that in the general case, the proofs cannot be based on Malliavin calculus as in [67]. Therefore, our proofs are mainly based on the Georgii-Nguyen-Zessin identity which characterizes the Papangelou conditional intensity.

Perhaps more interestingly, we obtain as a corollary the proper definition of the divergence operator, that is the adjoint in $\mathrm{L}^2(\mathcal{X},\mu)$ of the difference operator $D_xF(\xi)=F(\xi\cup x)-F(\xi)$. In the following, we also obtain an analogue of the Skohorod isometry which is well known for the Poisson point process, and which we generalize to a general point process.

Lastly, we study a random transformation of the point process, and calculate its correlation functions. The way we proceed is that we characterize the correlation functions as the unique functionals which verify our moment formulae, i.e. if the point process verifies a certain type of moment formula, we show that it necessarily has the correlation functions

that appear in the formula.

To conclude, we mention that the condition for the random transformation to correctly define a quasi-continuous point process that initially appeared in our calculations was the cyclicity condition mentioned in [67]. However, after discussions with J.-C. Breton and N. Privault, it came to our attention that this condition did not actually verify certain properties which were believed to be true. Therefore, we had to modify our approach in order to prove Theorem 6.3.1. Our proof is original and is largely based on the integration by parts formula for the difference operator. Since then, the previously mentioned authors have published the paper [10] which use novel proofs to prove some of the results of this chapter, as well as some new ones. The interested reader can thus find further results in this direction therein.

Appendix A

Disaster recovery algorithm

A.1 Introduction

In this first appendix, we present a homology based algorithm for disaster recovery of wireless networks. We represent wireless networks with Čech simplicial complexes characterizing their coverage. Given a set of vertices and their coverage radius, our algorithm first adds supernumerary vertices in order to patch every existing coverage hole and connect every components, then runs an improved version of the reduction algorithm presented in [86] in order to reach an optimal result with a minimum number of added vertices. At the end, we obtain the locations in which to put new nodes. For the addition of new vertices, we first compared three usual methods presenting low complexity: grid positioning, uniform positioning and the use of the Sobol sequence, a statistical tool built to provide uniform coverage of the unit square. Then, we propose the use of a determinantal point process: the Ginibre point process. This process has the ability to create repulsion between vertices, and therefore has the inherent ability to locate areas with low density of vertices: namely coverage holes. Therefore using this process, we will optimally patch the damaged wireless network. The use and simulation of determinantal point processes in wireless networks is new, and it provides tremendous results compared to classic methods. We finally compared our whole disaster recovery algorithm performance to the classic recovery algorithm performance: the greedy algorithm for the set cover problem.

This is the first algorithm that we know of that adds too many vertices then remove them to reach an optimal result instead of adding the exact needed number of vertices. This, first, allows flexibility in the choice of the new vertices positions, which can be useful when running the algorithm in a real life scenario. Indeed, in case of a disaster, every locations are not always available for installing new nodes and preferring some areas or locations can be done with our algorithm. The originality of our work lies also in the choice of the vertices addition method we suggest. On top of flexibility, our algorithm provides a more reliable repaired wireless network than other algorithms. Indeed, adding the exact needed number of vertices can be optimal mathematically speaking but it is very sensitive to the adherence of the nodes positions chosen by the algorithm. To compare our work to literature, we can see that the disaster recovery problem can be viewed as a set cover problem. It suffices to define the universe as the area to be covered and the subsets as the balls of radii the coverage radii. Then the question is to find the optimal set of subsets that cover the universe, considering there are already balls centered on the existing vertices. A greedy algorithm can solve this problem as explained in [16]. We can see in [32] that ϵ-nets also provide an algorithm for the set cover problem via a sampling of

the universe. We can also cite landmark-based routing, seen in [25] and [6], which, using furthest point sampling, provides a set of nodes for optimal routing that we can interpret as a minimal set of vertices to cover an area.

The remainder of this chapter is structured as follows: after a section on related work we present the main idea of our disaster recovery algorithm in Section A.3. Then in Section A.4, we compare usual vertices addition methods. In Section A.5, we expose the determinantal method for new vertices addition. Finally in Section A.6 we compare the performance of the whole disaster recovery algorithm with the greedy algorithm for the set cover problem.

A.2 Recovery in future cellular networks

The first step of recovery in cellular networks is the detection of failures. The detection of the failure of a cell occurs when its performance is considerably and abnormally reduced. In [57], the authors distinguish three stages of cell outage: degraded, crippled and catatonic. This last stage matches with the event of a disaster when there is complete outage of the damaged cells. After detection, compensation from other nodes can occur through relay assisted handover for ongoing calls, adjustments of neighboring cell sizes via power compensation or antenna tilt. In [5], the authors not only propose a cell outage management description but also describe compensation schemes. These steps of monitoring and detection, then compensation of nodes failures are comprised under the self-healing functions of future cellular networks described in [1].

In this work, we are interested in what happens when self-healing is not sufficient. In case of serious disasters, the compensation from remaining nodes and traffic rerouting might not be sufficient to provide service everywhere. In this case, the cellular network needs a manual intervention: the adding of new nodes to compensate the failures of former nodes. However a traditional restoration with brick-and-mortar base stations could take a long time, when efficient communication is particularly needed. In these cases, a recovery trailer fleet of base stations can be deployed by operators [56], it has been for example used by AT&T after 9/11 events. But a question remains: where to place the trailers carrying the recovery base stations. An ideal location would be adjacent to the failed node. However, these locations are not always available because of the disaster, and the recovery base stations may not have the same coverage radii than the former ones. Therefore a new deployment for the recovery base stations has to be decided, in which one of the main goal is complete coverage of damaged area. This becomes a mathematical set cover problem. It can been solved by a greedy algorithm [16], ϵ-nets [32], or furthest point sampling [6, 25]. But these mathematical solutions provide an optimal mathematical result that do not consider any flexibility at all in the choosing of the new nodes positions, and that can be really sensitive to imprecisions in the nodes positions.

A.3 Main idea

We consider a damaged wireless network presenting coverage holes with a fixed boundary, in order to know the domain to cover, of which we can see an example in Figure A.1.

We consider as inputs the set of existing vertices: the nodes of a damaged wireless

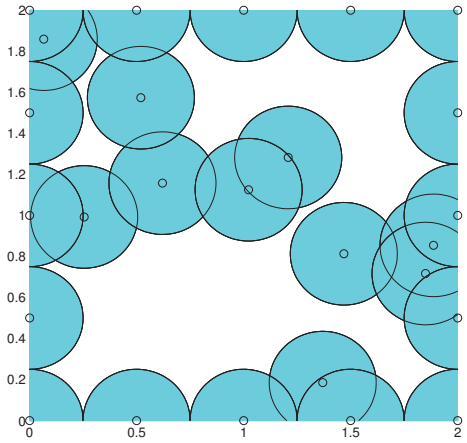


Figure A.1: A damaged wireless network with a fixed boundary.

network, and their coverage radii. We also need a list of boundary nodes, these nodes can be fictional, but we need to know the whole area that is to be covered. Then we build the Čech complex characterizing the coverage of the wireless network, the Betti number β_1 of the Čech complex counting the number of coverage holes of the wireless network. We restrict ourselves to wireless networks with a fixed communication radius r, but it is possible to build the Čech complex of a wireless networks with different coverage radii using the intersection of different size coverage balls.

The algorithm begins by adding new vertices in addition to the set of existing vertices presenting coverage holes. We suggest here the use of three usual methods, and the new determinantal addition method. As we can see in Section A.4, it is possible to consider any vertices addition methods must they be deterministic or random based: flexibility is one of the greatest advantage of our algorithm. In particular, it is possible to consider a method with pre-defined positions for some of the vertices in real-life scenarii.

For any non-deterministic method, we choose that the number of added vertices, that we denote by N_a, is determined as follows. First, it is set to be the minimum number of vertices needed to cover the whole area minus the number of existing vertices. This way, we take into account the number of existing vertices, that we denote by N_i. Then the Betti numbers β_0 and β_1 are computed via linear algebra thanks to the simplicial complex representation. If there is still more than one connected component, and coverage holes, then the number of added vertices N_a is incremented with a random variable u following an exponential growth:

— $N_a := \lceil \frac{a^2}{\pi r^2} \rceil - N_i$.
— After adding the N_a vertices, if $\beta_0 \neq 1$ or $\beta_1 \neq 0$,
  Then, $N_a = N_a + u$, and $u = 2 * u$.

The next step of our approach is to run the coverage reduction algorithm from [86] which maintains the topology of the wireless network: the algorithm removes vertices from the simplicial complex without modifying its Betti numbers. At this step, we remove some of the supernumerary vertices we just added in order to achieve an optimal result with a minimum number of added vertices.

We give in Algorithm 9 the outline of the algorithm. The algorithm requires the set

of initial vertices ω_i, the fixed coverage radius r, as well as the list of boundary vertices L_b. It is important to note that only connectivity information is needed to build the Čech complex.

Algorithm 9 Disaster recovery algorithm

Require: Set of vertices ω_i, radius r, boundary vertices L_b
  Computation of the Čech complex $X = \mathcal{C}_r(\omega_i)$
  $N_a = \lceil \frac{a^2}{\pi r^2} \rceil - N_i$
  Addition of N_a vertices to X following chosen method
  Computation of $\beta_0(X)$ and $\beta_1(X)$
  $u = 1$
  while $\beta_0 \neq 1$ or $\beta_1 \neq 0$ **do**
    $N_a = N_a + u$
    $u = 2 * u$
    Addition of N_a vertices to X following chosen method
    Computation of $\beta_0(X)$ and $\beta_1(X)$
  end while
  Coverage reduction algorithm on X
  return List L_a of kept added vertices.

A.4 Vertices addition methods

In this section, we propose three vertices addition methods. The aim of this part of the algorithm is to add enough vertices to patch the coverage of the simplicial complex, but the less vertices the better since the results will be closer to the optimal solution. We consider grid and uniform positioning which require minimum simulation capacities and are well known in wireless networks management. Then we propose the use of the Sobol sequence, which is a statistical tool built to provide uniform coverage of the unit square. The grid method is deterministic, so the number of added vertices as well as their position are set. The uniform method is random, the number of added vertices is then computed as presented in Section A.3.

A.4.1 Grid

The first method we suggest ensures perfect coverage: the new vertices are positioned along a square grid in a lattice graph where the distance between two neighboring vertices is $\sqrt{2}r$. The number of vertices is set. Therefore this method is completely independent from the initial configuration. We can see an example of the grid vertices addition method on the damaged network of Figure A.1 in Figure A.2. Existing vertices are black circles while added vertices are red plusses.

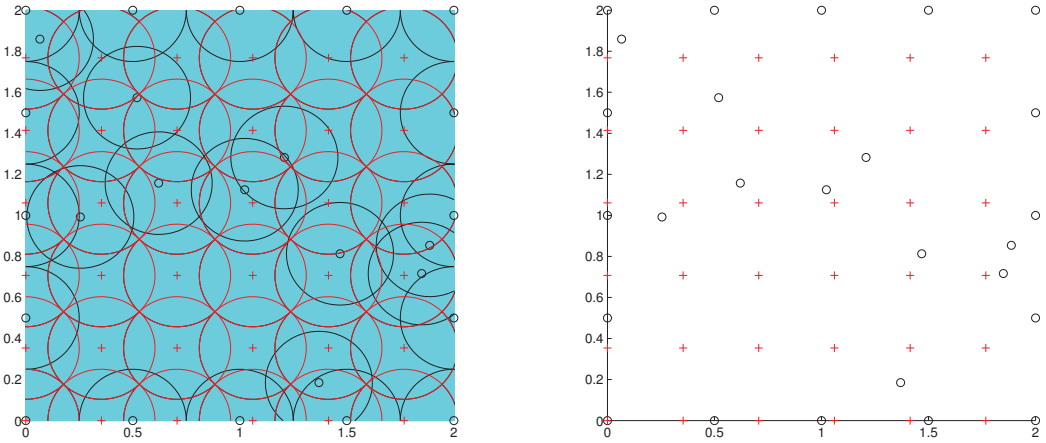


Figure A.2: With the grid addition method.

A.4.2 Uniform

Here, the number of added vertices N_a is computed accordingly to the method presented in Section A.3, taking into account the number of existing vertices N_i. Then the N_a vertices are sampled following a uniform law on the entire domain. An obtained configuration with this method on the network of Figure A.1 is shown in Figure A.3.

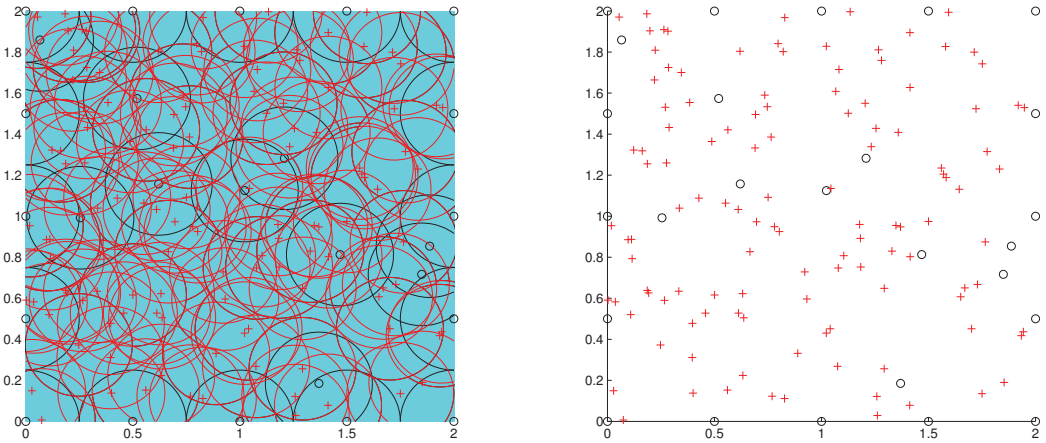


Figure A.3: With the uniform addition method.

A.4.3 Sobol sequence

Thanks to this method, we are able to take into account the positions of the new added vertices. The Sobol sequence is a statistical tool used to provide uniform coverage of the unit square. Thus, vertices positioned with the Sobol sequences reach complete coverage faster than uniform positioning because the aggregation phenomenon is statistically avoided. The Sobol initialization set is known, for instance on a square the first position is the middle of the square, then come the middles of the four squares included in the big

square, etc. To randomize the positions drawn, the points are scrambled. Therefore the complexity of the simulation of this method is really low.

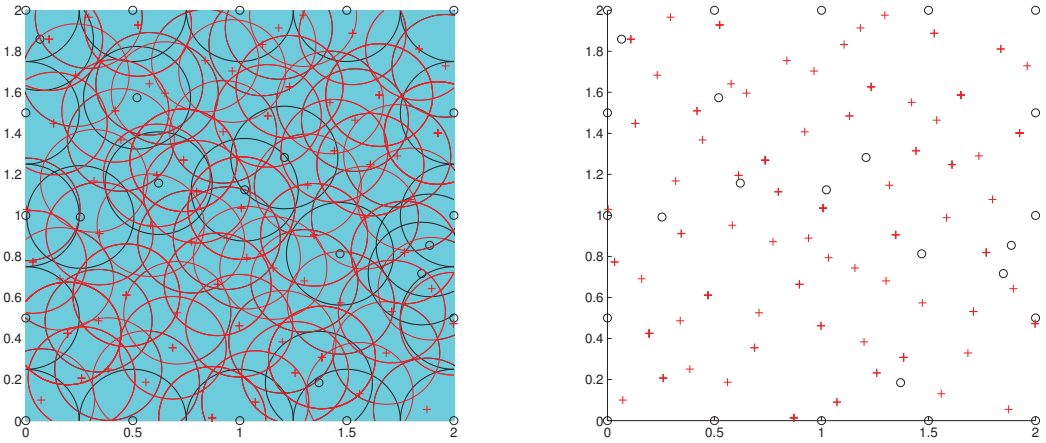


Figure A.4: With the Sobol sequence addition method.

For our simulation, we used the set of initialization numbers provided by Bratley and Fox in [9]. Then we scrambled the points produced with the random method described in [51]. An example of this method is given in Figure A.4.

A.4.4 Comparison

We can compare the vertices addition method presented here along two variables: their complexity and their efficiency. First, we compare the complexities of the two methods. They all are of complexity $O(N_a)$: computations of N_a positions, and the Sobol method scramble N_a positions already known by most simulation tools. For the random methods we have to add the complexity of computing the coverage via the Betti numbers, which is of the order of the number of triangles times the number of edges that is $O((N_a+N_i)^5(\frac{r}{a})^6)$ for a square of side a according to [19].

To compare the methods efficiency we count the number of vertices each have to add on average to reach complete coverage. The grid method being determinist, the number of added vertices is constant: $N_a = (\lfloor \frac{a}{\sqrt{2}r} \rfloor + 1)^2$ for a Čech complex or $N_a = (\lfloor \frac{a}{2r} \rfloor + 1)^2$ for a Vietoris-Rips complex which is an approximation of the Čech complex easier to simulate. We can see in Table A.1 the mean number of added vertices on 1000 simulations for each method in different scenarii on a square of side $a = 1$ with coverage radius $r = 0.25$, and a Vietoris-Rips complex. Scenarii are defined by the mean percentage of area covered before running the recovery algorithm: if there are many or few existing vertices, and thus few or many vertices to add. We need to note that number of added vertices is computed following our incrementation method presented in Section A.3 and these results only concern the vertices addition methods before the reduction algorithm runs.

We can observe that the Sobol sequence method gives better results than the uniform method except for a 60% of covered area. Since the Sobol sequence partitions uniformly the area, it takes advantages that there are not too many existing vertices in the scenarii

Percentage of area initially covered	20%	40%	60%	80%
Grid method	9.00	9.00	9.00	9.00
Uniform method	32.76	29.18	23.71	16.46
Sobol sequence method	29.80	29.04	24.16	15.98

Table A.1: Mean number of added vertices $\mathbb{E}N_a$

with small percentage of covered area. The grid method is mathematically optimal for the number of added vertices to cover the whole area, however it is not optimal in a real life scenario where positions can not be defined with such precision, and any imprecision leads to a coverage hole. This method fares even or better both in complexity and in number of added vertices.

A.5 Determinantal addition method

In this section, we present the determinantal method and compare it to the three methods presented in Section A.4.

A.5.1 Definitions

The most common point process in wireless network representation is the Poisson point process. However in this process, conditionally to the number of vertices, their positions are independent from each other (as in the uniform positioning method presented in Section A.4). This independence creates some aggregations of vertices, that is not convenient for our application. That is why we introduce the use of determinantal point processes, in which the vertices positions are not independent anymore. We can see in Figure A.5 the differences between points sampled uniformly and sampled with repulsion on the unit disk. We can see that the independence of vertices positions of the Poisson point process creates some clusters, while determinantal processes provide a more uniform coverage.

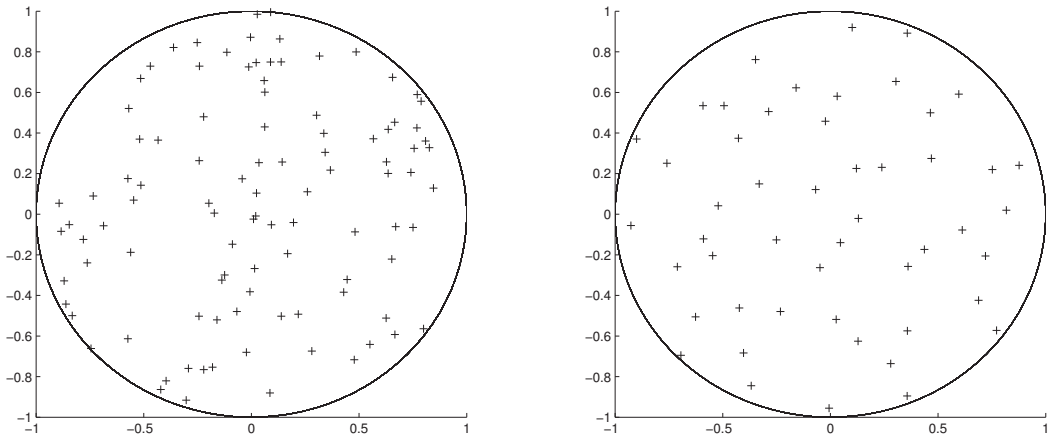


Figure A.5: Uniform vs determinantal sampling.

General point processes can be characterized by their so-called Papangelou intensity.

Informally speaking, for x a location, and ω a realization of a given point process, that is a set of vertices, $c(x,\omega)$ is the probability to have a vertex in an infinitesimal region around x knowing the set of vertices ω. For Poisson process, $c(x,\omega) = 1$ for any x and any ω. A point process is said to be repulsive (respectively attractive) whenever $c(x,\omega) \geq c(x,\zeta)$ (resp. $c(x,\omega) \leq c(x,\zeta)$) as soon as $\omega \subset \zeta$. For repulsive point process, that means that the greater the set of vertices, the smaller the probability to have an other vertex.

Among repulsive point processes, we are in particular interested in determinantal processes:

Definition 17 (Determinantal point process)**.** *Given X a Polish space equipped with the Radon measure μ, and K a measurable complex function on X^2, we say that N is a determinantal point process on X with kernel K if it is a point process on X with correlation functions $\rho_n(x_1,\ldots,x_n) = \det(K(x_i,x_j)_{1\leq i,j\leq n})$ for every $n \geq 1$ and $x_1,\ldots,x_n \in X$.*

We can see that when two vertices x_i and x_j tends to be close to each other for $i \neq j$, the determinant tends to zero, and so does the correlation function. That means that the vertices of N repel each other. There exist as many determinantal point processes as functions K. We are interested in the following:

Definition 18 (Ginibre point process)**.** *The Ginibre point process is the determinantal point process with kernel $K(x,y) = \sum_{k=1}^{\infty} B_k \phi_k(x)\overline{\phi_k(y)}$, where $B_k, k = 1,2,\ldots,$ are k independent Bernoulli variables and $\phi_k(x) = \frac{1}{\sqrt{\pi k!}} e^{\frac{-|x|^2}{2}} x^k$ for $x \in \mathbb{C}$ and $k \in \mathbb{N}$.*

The Ginibre point process is invariant with respect to translations and rotations, making it relatively easy to simulate on a compact set. Moreover, the repulsion induced by a Ginibre point process is of electrostatic type. The principle behind the repulsion lies in the probability density used to draw vertices positions. The probability to draw a vertex at the exact same position of an already drawn vertex is zero. Then, the probability increases with increasing distance from every existing vertices. Therefore the probability to draw a vertex is greater in areas the furthest away from every existing vertices, that is to say in coverage holes. Therefore, added vertices are almost automatically located in coverage holes thus reducing the number of superfluous vertices.

A.5.2 Simulation

Using determinantal point processes allows us to not only take into account the number of existing vertices, via the computation of N_a, but we also take into account the existing vertices positions, then every new vertex position as it is added. It suffices to consider the N_i existing vertices as the N_i first vertices sampled in the process, then each vertex is taken into account as it is drawn. The Ginibre process is usually defined on the whole plane thus we needed to construct a process with the same repulsive characteristics but which could be restricted to a compact set. Moreover, we needed to be able to set the number of vertices to draw. Due to space limitations, we will not delve into these technicalities but they are developed in Chapter 4. We can see a realization of our simulation for the recovery of the wireless network of Figure A.1 in Figure A.6.

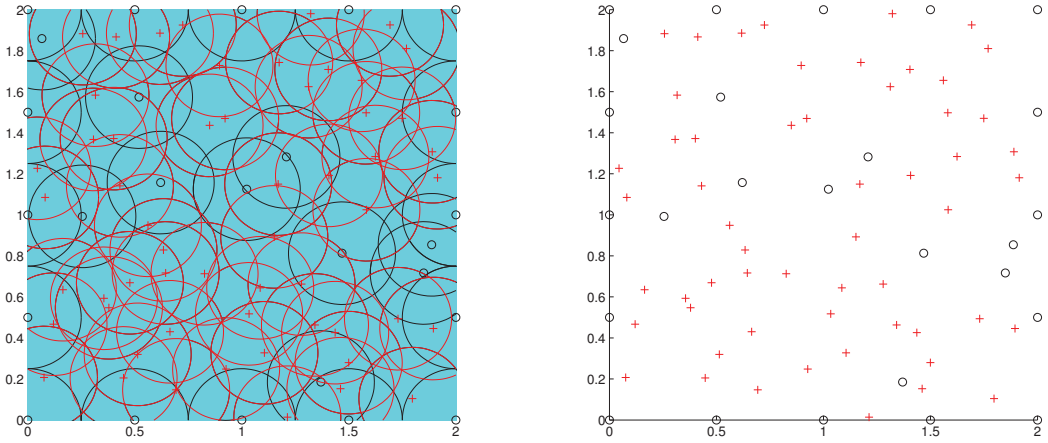


Figure A.6: With the determinantal addition method with a Ginibre basis.

A.5.3 Comparison

We now compare the determinantal vertices addition method to the methods presented in Section A.4.

As for the complexity, since the determinantal method takes into account the position of both existing vertices and randomly added vertices, it is the more complex. First taking into account the existing vertices positions is of complexity $O(N_i^2)$, then the position drawing with the rejection sampling is of complexity $O(N_a(N_a + N_i))$ at most. Thus we have a final complexity of $O(N_i^2 + N_a^2 + N_a N_i))$. To which we add the Betti numbers computation complexity: $O((N_a + N_i)^5(\frac{r}{a})^6)$.

We give in Table A.2 the comparison between the mean number of added vertices for the three methods. The simulation parameters being the same as Section A.4. The determinantal method provides the best results in all scenarii among the random methods by far. And it is the best method among all for the most covered scenario.

Percentage of area initially covered	20%	40%	60%	80%
Grid method	9.00	9.00	9.00	9.00
Uniform method	32.76	29.18	23.71	16.46
Sobol sequence method	29.80	29.04	24.16	15.98
Determinantal method	14.07	12.52	9.67	5.73

Table A.2: Mean number of added vertices $\mathbb{E}N_a$

A.6 Performance comparisons

After adding the new vertices, according to Algorithm 9, we run the coverage reduction algorithm described in [86]. Therefore, from the N_a added vertices we keep only what we call the final number of added vertices $N_f < N_a$. We can see in Figure A.7 an execution of the reduction algorithm on the intermediate configuration of Figure A.6. Removed vertices

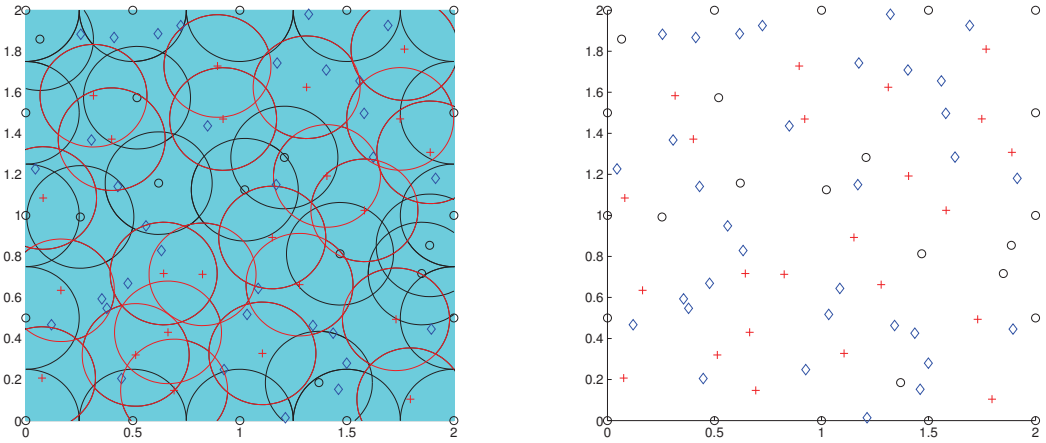


Figure A.7: The coverage reduction algorithm run on the determinantal method example.

are represented by blue diamonds. We now compare the performance results of the whole disaster recovery algorithm to the most known coverage recovery algorithm: the greedy algorithm for the set cover problem.

A.6.1 Complexity

The greedy algorithm method lays a square grid of parameter $\sqrt{2}r$ for the Čech complex of potential new vertices. Then the first added vertex is the furthest from all existing vertices. The algorithm goes on adding the furthest potential vertex of the grid from all vertices (existing+added). It stops when the furthest vertex is in the coverage ball of an existing or added vertex. Then for the $(i+1)$-th vertex addition, the greedy algorithm computes the distances from all $N_i + i$ existing vertices to all $(\lfloor \frac{a}{\sqrt{2}r} \rfloor + 1)^2 - i$ potential vertices. Therefore the complexity of the greedy algorithm is in $O((N_i+N_a)(\lfloor \frac{a}{\sqrt{2}r} \rfloor +1)^2))$.

For the complexity of our algorithm, we consider first the complexity of building the simplicial complex associated with the network which is in $O((N_i+N_a)^C)$, where C is the clique number. This complexity seems really high since C can only be upper bounded by $N_i + N_a$ in the general case but it is the only way to compute the coverage when vertices position are not defined along a grid. Then the complexity of the coverage reduction algorithm is of the order of $O((1+(\frac{r}{a})^2)^{N_i+N_a})$. So the greedy algorithm appears less complex than ours in the general case. However when r is small with respect to a or when the dimension is greater than 2, then the power factor becomes $d > 2$ and C is a finite integer, so the trend is reversed.

A.6.2 Mean final number of added vertices

We compare here the mean number of added vertices between our homology algorithm with the determinantal addition method, and the greedy algorithm.

Results presented in Table A.3 are simulated in the same conditions as in Section A.4 and given in mean over 1000 simulations. They concern the final number of added vertices: the number of added vertices kept after the reduction algorithm, or added with the greedy algorithm. It is important to note that our algorithm with the grid method gives the exact

same result as the greedy algorithm, number of added vertices and their positions being exactly the same.

Percentage of area initially covered	20%	40%	60%	80%
Greedy algorithm	3.69	3.30	2.84	1.83
Homology algorithm	4.42	3.87	2.97	1.78

Table A.3: Mean final number of added vertices $\mathbb{E}N_f$

The numbers of vertices added in the final state both with our recovery algorithm and the greedy algorithm are roughly the same. They both tend to the minimum number of vertices required to cover the uncovered area depending on the initial configuration. Nonetheless, we can see that our algorithm performs a little bit worse than the greedy algorithm in the less covered area scenarii because the vertices are not optimally positioned and it can be seen when just a small percentage of area is covered, and whole parts of the grid from the greedy algorithm are used, instead of isolated vertices. In compensation, our homology algorithm performs better in more covered scenarii.

A.6.3 Smoothed robustness

To show the advantages of our disaster recovery algorithm we choose to evaluate the robustness of the algorithm when the added vertices positions are slightly moved, i.e. when the nodes positioning does not strictly follow the theoretical positioning. In order to do this, we apply a Gaussian perturbation to each of the added vertices position. The covariance matrix of the perturbation is given by $\Sigma = \sigma^2 \mathrm{I}$ with $\sigma^2 = 0.01$, which means that the standard deviation for each vertex is of $\sigma = 0.1$. Other simulations parameters are unchanged, results in Table A.4 and A.5 are given in mean over 1000 simulations. First, we compute the average number of holes β_1 created by the Gaussian perturbation in Table A.4. Then in Table A.5, we counted the percentage of simulations in which the number of holes is still zero after the Gaussian perturbation on the new vertices positions.

Percentage of area initially covered	20%	40%	60%	80%
Greedy algorithm	0.68	0.65	0.45	0.35
Homology algorithm	0.62	0.53	0.37	0.26

Table A.4: Mean number of holes $\mathbb{E}\beta_1$ after the Gaussian perturbation

Percentage of area initially covered	20%	40%	60%	80%
Greedy algorithm	40.8%	47.7%	61.0%	69.3%
Homology algorithm	50.9%	58.1%	67.9%	75.3%

Table A.5: Probability that there is no hole $\mathbb{P}(\beta_1 = 0)$ after the Gaussian perturbation

We can see that the perturbation on the number of holes decreases with the percentage of area initially covered, since the initial vertices are not perturbed. Our homology algorithm clearly performs better, even in the least covered scenarii, there are less than 50% of simulations that create coverage holes, which is not the case for the greedy algorithm.

The greedy algorithm also always create more coverage holes in mean than our disaster recovery algorithm for the same vertices positions perturbation. Therefore our algorithm seems more fitted to the disaster recovery case when a recovery network is deployed in emergency both indoor, via Femtocells, and outdoor, via a trailer fleet, where exact GPS locations are not always available, and exact theoretical positioning is not always followed.

Appendix B

Stochastic dynamics of determinantal processes by integration by parts

B.1 Introduction

Determinantal processes are point processes that exhibit a repulsion property, and were introduced to represent the configuration of fermions, cf. [50], [78] and [82]. They are known to be connected with the zeros of analytic functions, cf. [35] and references therein.

In this paper we construct Dirichlet forms related to determinantal processes, and we apply them to derive the existence of the associated interacting diffusion processes. For this we provide an integration by parts formula for functionals of determinantal processes which is based on a quasi-invariance result proved in [14]. This integration by parts formula is extended to closed gradient and divergence operators thanks to the use of a set of test functionals different from the one considered in [14], cf. (5.5) and Theorem 5.3.2-(i) and (ii). Our approach follows the lines of [2] and our construction differs from the one considered in [83] which is based on sample-path identities. Such a construction can be applied to derive formulas for density estimation and sensitivity analysis for functionals of determinantal processes along the lines of [69].

Our main result, Theorem B.4.1, provides the symmetric Dirichlet form associated to a determinantal process. An application to the associated stochastic dynamics follows in Theorem B.5.1, in which we prove the existence of the diffusion process associated to a determinantal process satisfying the assumptions of Theorem 5.3.2.

We proceed as follows. In Section 5.2 we recall the definitions of point processes and determinantal process, based on [17], [18], [55] and [85] for point processes, and on [35] for determinantal processes. We also refer to [11], [21] and [79] for the required background on functional analysis.

B.2 Preliminaries

B.2.1 Locally finite point processes

Let (S, d_S) be a locally compact Polish space, and denote by $\mathcal{B}_S$ the associated Borel σ-algebra. For any subset $B \subseteq S$, let $\sharp B$ denote the cardinality of B, setting $\sharp B = \infty$ if B

is not finite. We denote by N_{lf} the set of locally finite point configurations on S:

$$\mathrm{N}_{lf}:=\{B \subseteq S\colon \sharp(B \cap D) < \infty, \text{ for any compact } D \subseteq S\},$$

equipped with the σ-field

$$\mathcal{N}_{lf}:=\sigma(\{B \in \mathrm{N}_{lf} \ : \ \sharp(B \cap D) = m\},\, m \geq 0,\, D \subseteq S \text{ compact}).$$

We define similarly N_f the set of finite point configurations on S:

$$\mathrm{N}_f:=\{B \subseteq S\colon \sharp B < \infty\},$$

N_f is naturally equipped with the trace σ-field $\mathcal{N}_f = \mathcal{N}_{lf}|_{\mathrm{N}_f}$. Lastly, for any compact $D \subseteq S$, let N_f^D be the space of finite configurations on D, and $\mathcal{N}_f^D$ the associated σ-field. As in [28], we define for any Radon measure μ on $(S, \mathcal{B}_S)$ the μ-sample measure L^μ on $(\mathrm{N}_f, \mathcal{N}_f)$ by

$$\int_{\mathrm{N}_f} f(\alpha)\, L^\mu(\mathrm{d}\alpha) := \sum_{n\geq 0} \frac{1}{n!} \int_{S^n} f(\{x_1, \ldots, x_n\})\, \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_n), \tag{B.2.1}$$

for any measurable $f : \mathrm{N}_f \to \mathbb{R}$.
Let $\mathbf{X}$ be a locally finite point process on S, i.e. a measurable mapping defined on some probability space $(\Omega, \mathcal{F}, P)$ and taking values on $(\mathrm{N}_{lf}, \mathcal{N}_{lf})$. We will scarcely denote by $\mathbf{X}$ an element of N_{lf}, omitting the fact that $\mathbf{X}$ is in fact a mapping. Moreover, we assume that $\mathbf{X}$ is *simple*, i.e. $\mathbf{X}(\{x\}) \in \{0, 1\}$ almost surely (a.s.), for all $x \in S$, where $\mathbf{X}(B)$ denotes the number of points of $\mathbf{X}$ on $B \subseteq S$, i.e. $\mathbf{X}(B) := \sharp(\mathbf{X} \cap B)$. We also denote by

$$\mathbf{X}^D = \mathbf{X} \cap D = \{X_1, \ldots, X_{\mathbf{X}(D)}\}$$

the restriction to D of the point process $\mathbf{X} \equiv \{X_n\}_{1 \leq n \leq \mathbf{X}(S)}$, where the previous notation includes the case $\mathbf{X}(S) = \infty$. In the following, we will denote by P the law of $\mathbf{X}$ and by P_D the law of $\mathbf{X}^D$.

The correlation functions of $\mathbf{X}$, with respect to (*w.r.t.*) a Radon measure μ on $(S, \mathcal{B}_S)$, are (if they exist) measurable functions $\rho_k : S^k \longrightarrow [0, \infty)$ such that

$$\mathrm{E}\left[\prod_{i=1}^{k} \mathbf{X}(B_i)\right] = \int_{B_1 \times \ldots \times B_k} \rho_k(x_1, \ldots, x_k)\, \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_k),$$

for any family of mutually disjoint subsets $B_1, \ldots, B_k$ of S, $k \geq 1$. We require in addition that $\rho_k(x_1, \ldots, x_k) = 0$ whenever $x_i = x_j$ for some $1 \leq i \neq j \leq k$. When ρ_1 exists, the measure $\rho_1(x)\, \mu(\mathrm{d}x)$ is known as the intensity measure of $\mathbf{X}$.

For any compact subset $D \subseteq S$, the Janossy densities of $\mathbf{X}$, *w.r.t.* μ are (if they exist) measurable functions $j_D^n : D^n \to \mathbb{R}$ satisfying, for all measurable functions $f : \mathrm{N}_f^D \to \mathbb{R}$,

$$\mathrm{E}\left[f(\mathbf{X}^D)\right] = \sum_{n\geq 0} \frac{1}{n!} \int_{D^n} f(\{x_1, \ldots, x_n\})\, j_D^n\,(x_1, \cdots, x_n)\ \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_n), \tag{B.2.2}$$

i.e. j_D is the density of P_D with respect to L_D^μ (the restriction to N_D^f of L^μ), when $P_D \ll L_D^\mu$. We remark that we shall sometimes use the simplified notation $j_D(\mathbf{x}) := j_D^{\mathbf{x}(D)}(x_1, \ldots, x_{\mathbf{x}(D)})$.

B.2.2 Kernels and integral operators

Let μ be a Radon measure on $(S, \mathcal{B}_S)$. For any compact set $D \subseteq S$, we denote by $\mathrm{L}^2(D,\mu)$ the Hilbert space of complex-valued square integrable functions *w.r.t.* the restriction of the Radon measure μ on D, equipped with the inner product

$$\langle f, g\rangle_{\mathrm{L}^2(D,\mu)} := \int_D f(x)\overline{g(x)}\,\mu(\mathrm{d}x), \quad f, g \in \mathrm{L}^2(D,\mu)$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. By definition, a kernel K is a measurable function from S^2 to $\mathbb{C}$. We say that K is locally square integrable if, for any compact $D \subseteq S$, we have

$$\int_{D^2} |K(x,y)|^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty. \tag{B.2.3}$$

To any locally square integrable kernel K, we associate the integral operator $\mathcal{K}_D : \mathrm{L}^2(D,\mu) \longrightarrow \mathrm{L}^2(D,\mu)$, where D is a compact subset of S, defined for $f \in L^2(D,\mu)$ by

$$\mathcal{K}_D f(x) := \int_D K(x,y) f(y)\,\mu(\mathrm{d}y), \quad \text{for } \mu\text{-almost all } x \in D.$$

A straightforward application of the Cauchy-Schwarz inequality shows that the operator $\mathcal{K}_D$ is bounded. In fact, it can be shown that $\mathcal{K}_D$ is a compact operator.

To any locally square integrable kernel K, we associate the integral operator $\mathcal{K}$ defined by

$$\mathcal{K} f(x) := \int_S K(x,y) f(y)\,\mu(\mathrm{d}y), \quad \text{for } \mu\text{-almost all } x \in S$$

for functions $f \in \mathrm{L}^2(S,\mu)$ that vanish outside a compact subset of S. Letting $\mathcal{P}_D$ denote the projection operator from $\mathrm{L}^2(S,\mu)$ onto $\mathrm{L}^2(D,\mu)$, we have $\mathcal{K}_D = \mathcal{P}_D \mathcal{K} \mathcal{P}_D$ and set $K_D(x,y) := \mathbf{1}_D(x) K(x,y) \mathbf{1}_D(y)$, for $x, y \in S$. The operator $\mathcal{K}$ is said to be Hermitian or self-adjoint if

$$K(x,y) = \overline{K(y,x)}, \quad \text{for } \mu^{\otimes 2}\text{-almost all } (x,y) \in S^2. \tag{B.2.4}$$

Equivalently, this means that the integral operators $\mathcal{K}_D$ are self-adjoint for any compact $D \subseteq S$. If $\mathcal{K}_D$ is self-adjoint, by the spectral theorem for self-adjoint and compact operators we have that $\mathrm{L}^2(D,\mu)$ has an orthonormal basis $\{\varphi_j^D\}_{j\geq 1}$ of eigenfunctions of $\mathcal{K}_D$. The corresponding eigenvalues $\{\lambda_j^D\}_{j\geq 1}$ have finite multiplicity (except possibly the zero eigenvalue) and the only possible accumulation point of the eigenvalues is the zero eigenvalue. Then, the kernel K_D of $\mathcal{K}_D$ can be written as

$$K_D(x,y) = \sum_{j\geq 1} \lambda_j^D \varphi_j^D(x) \overline{\varphi_j^D(y)}, \tag{B.2.5}$$

for $x, y \in D$. We say that an operator $\mathcal{K}$ is positive (respectively non-negative) if its spectrum is included in $(0, +\infty)$ (respectively $[0, +\infty)$). For two operators $\mathcal{K}$ and $\mathcal{I}$, we will say that $\mathcal{K} > \mathcal{I}$ (respectively $\mathcal{K} \geq \mathcal{I}$) in the operator ordering if $\mathcal{K} - \mathcal{I}$ is a positive operator (respectively non-negative operator).

We say that a self-adjoint integral operator $\mathcal{K}_D$ is of trace class if

$$\sum_{j\geq 1} |\lambda_j^D| < \infty.$$

We will then define the trace of the operator $\mathcal{K}_D$ as $\mathrm{Tr}\,\mathcal{K}_D = \sum_{j\geq 1}\lambda_j^D$. If $\mathcal{K}_D$ is of trace class for every compact subset $D \subseteq S$, then we say that $\mathcal{K}$ is locally of trace class. It is easily seen that $\mathcal{K}^n$ is of trace class, for all $n \geq 2$, if $\mathcal{K}$ is of trace class. We also have the inequality $\mathrm{Tr}(\mathcal{K}^n) \leq \|\mathcal{K}\|^{n-1}\mathrm{Tr}(\mathcal{K})$, where $\|\cdot\|$ denotes the operator norm. Finally, we define the Fredholm determinant of $\mathbf{Id} + \mathcal{K}$, where $\|\mathcal{K}\| < 1$, as

$$\mathrm{Det}(\mathbf{Id} + \mathcal{K}) = \exp\left(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\mathrm{Tr}(\mathcal{K}^n)\right). \tag{B.2.6}$$

Here, $\mathbf{Id}$ denotes the identity operator on $\mathrm{L}^2(S,\mu)$.

B.2.3 Determinantal point processes on (S, d_S)

Throughout this paper we shall work under the following hypothesis:
(H1): *The operator $\mathcal{K}$ is locally of trace class, satisfies (3.3), and its spectrum is contained in $[0,1)$, i.e. $0 \leq \mathcal{K} \leq \mathbf{Id}$ and $\|\mathcal{K}\| < 1$.*

Suppose that $\mathcal{K}$ satisfies (**H1**). A locally finite and simple point process $\mathbf{X} \equiv \{X_n\}_{1\leq n\leq \mathbf{X}(S)}$ on S is called a determinantal process if its correlation functions *w.r.t.* the Radon measure μ on S exist and satisfy

$$\rho_k(x_1,\ldots,x_k) = \det(K(x_i,x_j))_{1\leq i,j\leq k},$$

for any $k \geq 1$ and $x_1,\ldots,x_k \in S$. It is worth noting that, under **(H1)**, we can choose a proper kernel for $\mathcal{K}$, in the sense of Lemma A.4 in [28], and we have $\rho_k(x_1,\ldots,x_k) \geq 0$ for $\mu^{\otimes k}$ -a.e. $(x_1,\ldots,x_k) \in S^k$.

Existence and uniqueness (in law) of determinantal processes is guaranteed under **(H1)** by the results in [50], [78] and [80]. See also Lemma 4.2.6 and Theorem 4.5.5 in [35]. More precisely, if a kernel K and its associated integral operator $\mathcal{K}$ satisfy (**H1**), then there exists a determinantal process $\mathbf{X}$ on S with kernel K. Moreover, for any compact $D \subseteq S$ there exist constants $c_1(D), c_2(D) > 0$ such that $P(\mathbf{X}(D) > k) \leq c_1(D)\mathrm{e}^{-c_2(D)k}$ for all $k \geq 1$, and in this case the correlation functions $\rho_k(x_1,\ldots,x_k)$ uniquely determine the law of the process. This is because of the fact that for disjoint compacts $D_1,\ldots,D_k \subseteq S$, the random vector $(\mathbf{X}(D_1),\ldots,\mathbf{X}(D_k))$ has a convergent Laplace transform in a neighborhood of zero, cf. [35] Remark 1.2.4.

Let $\mathcal{K}$ be an operator satisfying the assumption (**H1**). We define the operator $\mathcal{J}[D]$ on $\mathrm{L}^2(D,\mu)$ by

$$\mathcal{J}[D] := (\mathbf{Id} - \mathcal{K}_D)^{-1}\mathcal{K}_D.$$

The operator $\mathcal{J}[D]$ is called the local interaction operator and we emphasize the fact that unlike $\mathcal{K}_D$, $\mathcal{J}[D]$ is not a projection operator i.e., in general, $\mathcal{J}[D] \neq \mathcal{P}_D(I-\mathcal{K})^{-1}\mathcal{K}\mathcal{P}_D$. However, $\mathcal{J}[D]$ has some notable properties which are summarized in [28]. Let us state the ones that are useful to our purposes. First, $\mathcal{J}[D]$ is a bounded integral operator and, letting $J[D]$ denote its kernel, as a consequence of (3.4), we have

$$J[D](x,y) = \sum_{j\geq 0}\frac{\lambda_j^D}{1-\lambda_j^D}\,\varphi_j^D(x)\overline{\varphi_j^D(y)}, \tag{B.2.7}$$

for $x, y \in D$. Second, since $\mathcal{J}[D] \leq (1 - \|\mathcal{K}\|)^{-1}\mathcal{K}_D$, we have that $\mathcal{J}[D]$ is a trace class operator. For $\mathbf{x} = \{x_1, \ldots, x_n\} \in \mathrm{N}_f^D$, we denote by $\det J[D](\mathbf{x}) = \det J[D](\{x_1, \ldots, x_n\})$ the determinant $\det\left(J[D](x_i, x_j)\right)_{1 \leq i,j \leq n}$. Note that the function

$$(x_1, \ldots, x_n) \mapsto \det J[D](\{x_1, \ldots, x_n\})$$

is $\mu^{\otimes n}$-a.e. non-negative (thanks to Lemma A.4 in [28] for example) and symmetric in $x_1, \ldots, x_n$ (see e.g. [28]), and we simply write $\det J[D](\{x_1, \ldots, x_n\}) = \det J[D](x_1, \ldots, x_n)$. The local interaction operator is related to the Janossy density of a determinantal process, due to the following proposition:

Proposition B.2.1 ([78]). *Assume that the kernel $\mathcal{K}$ satisfies* **(H1)**. *Then, given a compact $D \subseteq S$ and $n \in \mathbb{N}^*$, the determinantal process $\mathbf{X}$ admits Janossy densities*

$$j_D^n(\mathbf{x}) = \mathrm{Det}(\mathbf{Id} - \mathcal{K}_D) \det J[D](\mathbf{x}), \tag{B.2.8}$$

where $\mathbf{x} = \{x_1, \ldots, x_n\} \in \mathrm{N}_f^D$. Moreover, it holds the following identity for the hole probability: $j_D^0(\emptyset) = \mathrm{Det}(\mathbf{Id} - \mathcal{K}_D)$.

Let us now define the operator $\mathcal{J} := (\mathbf{Id} - \mathcal{K})^{-1}\mathcal{K}$ which can be thought of as a global interaction operator. As proved in [28], $\mathcal{J}$ satisfies the expected properties: it is a bounded integral operator, locally trace class and its kernel $J(x, y)$ can be chosen as in Lemma A.4 of [28]. Furthermore, the determinantal process $\mathbf{X}$ is stochastically dominated by a Poisson process $\mathbf{Y}$ with intensity measure $J(x, x)\,\mu(\mathrm{d}x)$ (defined previously), i.e

$$\mathrm{E}[f(\mathbf{X})] \leq \mathrm{E}[f(\mathbf{Y})] \tag{B.2.9}$$

for any measurable function $f : \mathrm{N}_{lf} \to \mathbb{R}$ such that the expectations exist and $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \subset \mathbf{y}$.

B.3 Differential calculus and integration by parts

Hereafter we assume that S is a domain of $\mathbb{R}^d$, d_S is the Euclidean distance and $D \subset S$ is a fixed compact. We denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^d$ and by $x^{(i)}$ the i-th component of $x \in \mathbb{R}^d$.

B.3.1 Differential calculus

We denote by $\mathcal{C}^\infty(D, \mathbb{R}^d)$ the set of all $\mathcal{C}^\infty$-vector fields $v : D \longrightarrow \mathbb{R}^d$ and by $\mathcal{C}^\infty(D^k)$ the set of all $\mathcal{C}^\infty$-functions on D^k.

Definition 1. *A random variable (r.v.) $F(\mathbf{X}^D)$ is said to be in $\mathcal{S}_D$ if*

$$F(\mathbf{X}^D) = f_0 \mathbf{1}_{\{\mathbf{X}(D)=0\}} + \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} f_k(X_1, \ldots, X_k), \tag{B.3.1}$$

where $n \geq 1$ is an integer, for any $k = 1, \ldots, n$, $f_k \in \mathcal{C}^\infty(D^k)$ is a symmetric function and $f_0 \in \mathbb{R}$ is a constant.

Then, the following lemma holds:

Lemma B.3.1. *$\mathcal{S}_D$ is dense in $\mathrm{L}^2_D = \mathrm{L}^2(\mathrm{N}_f^D, P_D)$.*

Proof. Let $V \in \mathrm{L}^2_D$ and assume that $\mathrm{E}[G(\mathbf{X}^D)V] = 0$ for all $G(\mathbf{X}^D) \in \mathcal{S}_D$. We need to show that $V = 0$ a.s.. Since V is $\mathcal{N}_f^D$-measurable we have that $V = u(\mathbf{X}(D), X_1, \ldots, X_{\mathbf{X}(D)})$, for some measurable function u. Therefore, for all integers $n \geq 1$ and $G(\mathbf{X}^D) \in \mathcal{S}_D$, with a little abuse of notation, we get

$$G(\emptyset)u(0,\emptyset)P(\mathbf{X}(D)=0) = 0 \quad \text{and} \quad \mathrm{E}[G(X_1,\ldots,X_n)u(n,X_1,\ldots,X_n)\,|\,\mathbf{X}(D)=n] = 0. \tag{B.3.2}$$

The first equality above yields $u(0,\emptyset) = 0$. We now prove that, for any $n \geq 1$, $u(n, x_1, \ldots, x_n) = 0$ a.s. *w.r.t.* the probability measure, say $\pi_D^{(n)}$, with density

$$j_D^{(n)}(x_1,\ldots,x_n)\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_n)$$

(here, $j_D^{(n)}$ are the Janossy densities of $\mathbf{X}^D$ divided by $P(\mathbf{X}(D)=n)$.)

Denote by u_n^+ and u_n^- the positive and the negative part of $u(n,\cdot)$, respectively. Clearly (take $G \equiv 1$ in the second equality in (B.3.2)) we have

$$E_n := \mathrm{E}[u_n^+(X_1,\ldots,X_n)\,|\,\mathbf{X}(D)=n] = \mathrm{E}[u_n^-(X_1,\ldots,X_n)\,|\,\mathbf{X}(D)=n].$$

If $E_n = 0$ then $u_n^+ = u_n^- = 0$ $\pi_D^{(n)}$-a.s., hence $u_n = 0$ $\pi_D^{(n)}$-a.s.. If $E_n > 0$, then consider the probability measures on D^n:

$$\pi_D^{(n)\pm}(\mathrm{d}x_1\ldots\mathrm{d}x_n) := \widehat{u}_n^{\pm}(x_1,\ldots,x_n)\,\pi_D^{(n)}(\mathrm{d}x_1,\ldots,\mathrm{d}x_n),$$

where

$$\widehat{u}_n^{\pm}(x_1,\ldots,x_n) := \frac{1}{E_n}u_n^{\pm}(x_1,\ldots,x_n).$$

Let R be a rectangular cell in $\mathbb{R}^{dn}$ and take $G(x_1,\ldots,x_n) = \varphi_l(x_1,\ldots,x_n)$ where φ_l is a sequence in $\mathcal{C}_c^\infty(\mathbb{R}^{dn})$ such that $\varphi_l(x_1,\ldots,x_n)$ converges to $\mathbf{1}_{\{(x_1,\ldots,x_n)\in\mathrm{R}\}}$ as l goes to infinity, for all $x_1,\ldots,x_n$. Combining the second equality in (B.3.2) with the dominated convergence theorem, we have $\pi_D^{(n)+}(\mathrm{R}\cap D^n) = \pi_D^{(n)-}(\mathrm{R}\cap D^n)$. Therefore, $\pi_D^{(n)+} \equiv \pi_D^{(n)-}$ on the Borel σ-field $\mathcal{B}(D^n)$. So $u_n^+(x_1,\ldots,x_n) = u_n^-(x_1,\ldots,u_n)$ $\pi_D^{(n)}$-a.s., and the claim follows. □

The gradient of $F(\mathbf{X}^D) \in \mathcal{S}_D$ as in (5.5) is defined by

$$\nabla_x^{\mathrm{N}_{lf}} F(\mathbf{X}^D) := \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^k \mathbf{1}_{\{X_i\}}(x)\nabla_x f_k(X_1,\ldots,X_k), \qquad x \in D, \tag{B.3.3}$$

where ∇_x denotes the usual gradient on $\mathbb{R}^d$ with respect to the variable $x \in D$. For $v \in \mathcal{C}^\infty(D,\mathbb{R}^d)$, we also let

$$\nabla_v^{\mathrm{N}_{lf}} F(\mathbf{X}^D) := \sum_{k=1}^{\mathbf{X}(D)} \nabla_{X_k}^{\mathrm{N}_{lf}} F(\mathbf{X}^D)\cdot v(X_k) = \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^k \nabla_{X_i} f_k(X_1,\ldots,X_k)\cdot v(X_i), \tag{B.3.4}$$

where $\cdot$ denotes the inner product on $\mathbb{R}^d$.

B.3.2 Integration by parts and closability

In this section we give an integration by parts formula for determinantal processes, based on closed gradient and divergence operators. The proof relies on an integration by parts formula on the set of test functionals $\mathcal{S}_D$ introduced in (5.5), extending and making more precise the arguments of Theorem 10 page 289 of [14] and its proof.

(**H3**) : *We suppose that, for any $n \geq 1$, the function*

$$(x_1, \ldots, x_n) \longmapsto \det J[D](x_1, \ldots, x_n)$$

is continuously differentiable on the whole D^n.

Assuming that (**H1**) and (**H3**) hold, the potential energy is the function $U : \mathrm{N}_f^D \longrightarrow \mathbb{R}$ defined by

$$U[D](\mathbf{x}) := -\log \det J[D](\mathbf{x}).$$

We insist that since $\det J[D](\mathbf{x}) > 0$ for P-a.e. $\mathbf{x} \in \mathrm{N}_f^D$, U is well defined for P-a.e. $\mathbf{x} \in \mathrm{N}_f^D$.
We set

$$\begin{aligned} \nabla_v^{\mathrm{N}_{lf}} U[D](\mathbf{X}^D) &:= -\sum_{k=1}^{\infty} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} \frac{\nabla_{x_i} \det J[D](X_1, \ldots, X_k)}{\det J[D](X_1, \ldots, X_k)} \cdot v(X_i) \\ &= \sum_{k=1}^{\infty} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} U_{i,k}(X_1, \ldots, X_k) \cdot v(X_i), \end{aligned} \tag{B.3.5}$$

for any vector field $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$.

Under Conditions (**H1**) and (**H2**) we define the vector field

$$\beta^\mu(x) := \frac{\nabla \rho(x)}{\rho(x)},$$

as well as the random variable

$$B_v^\mu(\mathbf{X}^D) := \sum_{k=1}^{\mathbf{X}(D)} (-\beta^\mu(X_k) \cdot v(X_k) + \operatorname{div} v(X_k)), \qquad v \in \mathcal{C}^\infty(D, \mathbb{R}^d),$$

where div denotes the adjoint of the gradient ∇ on D, i.e.

$$\int_D g(x) \operatorname{div} \nabla f(x) \, \mathrm{d}x = \int_D \nabla f(x) \cdot \nabla g(x) \, \mathrm{d}x, \quad f, g \in \mathcal{C}^\infty(D).$$

Lemma B.3.2. *Assume that* (**H1**), (**H2**) *and* (**H3**) *hold. Then, for any $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \mathcal{S}_D$ and vector field $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$, we have*

$$\mathrm{E}[G(\mathbf{X}^D) \nabla_v^{\mathrm{N}_{lf}} F(\mathbf{X}^D)] = \mathrm{E}[F(\mathbf{X}^D) \nabla_v^{\mathrm{N}_{lf}*} G(\mathbf{X}^D)], \tag{B.3.6}$$

where

$$\nabla_v^{\mathrm{N}_{lf}*} G(\mathbf{X}^D) := -\nabla_v^{\mathrm{N}_{lf}} G(\mathbf{X}^D) + G(\mathbf{X}^D) \left(-B_v^\mu(\mathbf{X}^D) + \nabla_v^{\mathrm{N}_{lf}} U[D](\mathbf{X}^D) \right).$$

Proof. For any vector field $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$, consider the flow $\phi_t^v : D \longrightarrow D$, $t \in \mathbb{R}$, where for a fixed $x \in D$, the curve $t \mapsto \phi_t^v(x)$ is defined as the solution to the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^v(x) = v(\phi_t^v(x)), \quad \phi_0^v(x) = x.$$

For a given family of flows $\{\phi_t^v : t \in \mathbb{R}, v \in \mathcal{C}^\infty(D, \mathbb{R}^d)\}$, we define the mapping $\Phi_t^v : \mathrm{N}_f^D \longrightarrow \mathrm{N}_f^D$ by

$$\Phi_t^v(\mathbf{x}) := \{\phi_t^v(x) : x \in \mathbf{x}\}.$$

Following [2], for a functional $R(\mathbf{X}^D)$ of the determinantal process, we define the gradient $\nabla_v^{\mathrm{N}_{lf}} R(\mathbf{X}^D)$ as the directional derivative along v, i.e.

$$\nabla_v^{\mathrm{N}_{lf}} R(\mathbf{X}^D) := \frac{\mathrm{d}}{\mathrm{d}t} R(\Phi_t^v(\mathbf{X}^D))\Big|_{t=0},$$

provided the derivative exists. It is easy to check that formulas (B.3.4) and (B.3.5) are consistent with this definition. Note that as observed previously, the image measure $\mu \circ \phi_{-t}^v$ is absolutely continuous with respect to μ on D, with Radon-Nikodym derivative

$$\frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \operatorname{Jac}^{\phi_{-t}^v}(x),$$

where $\operatorname{Jac}^{\phi_{-t}^v}$ denotes the Jacobian of ϕ_{-t}^v. Note also that

$$\operatorname{Jac}^{\phi_t^v}(x) = \exp\left(-\int_0^t \operatorname{div} v(\phi_z^v(x))\,\mathrm{d}z\right) \tag{B.3.7}$$

and therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \operatorname{Jac}^{\phi_{-t}^v}(x)\right) = &- \exp\left(-\int_0^t \operatorname{div} v(\phi_z^v(x))\,\mathrm{d}z\right)\Big[\frac{\nabla\rho(\phi_{-t}^v(x))}{\rho(x)} \cdot v(\phi_{-t}^v(x)) \\ &+ \frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \operatorname{div} v(\phi_t^v(x))\Big]. \end{aligned} \tag{B.3.8}$$

Using the quasi-invariance property of determinantal processes discussed in the previous section, for any $t \in \mathbb{R}$ and $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \mathcal{S}_D$, we have

$$\mathrm{E}[F(\Phi_t^v(\mathbf{X}^D))G(\mathbf{X}^D)] \tag{B.3.9}$$

$$= \mathrm{E}\Big[F(\mathbf{X}^D)G(\Phi_{-t}^v(\mathbf{X}^D)) \left(\prod_{k=1}^{\mathbf{X}(D)} \frac{\rho(\phi_{-t}^v(X_k))}{\rho(X_k)} \operatorname{Jac}^{\phi_{-t}^v}(X_k)\right) \frac{\det J[D](\phi_{-t}^v(X_1), \ldots, \phi_{-t}^v(X_{\mathbf{X}(D)}))}{\det J[D](X_1, \ldots, X_{\mathbf{X}(D)})}\Big]. \tag{B.3.10}$$

We start by exchanging the derivative $\mathrm{d}/\mathrm{d}t$ with the expectation sign E in the above relation, for all $t \in I_0$ a neighborhood of zero. This interchange will be justified by integrability after (B.3.14) below. In this case we have

$$\mathrm{E}\left[G(\mathbf{X}^D)\frac{\mathrm{d}}{\mathrm{d}t}F(\Phi_t^v(\mathbf{X}^D))\right]$$

$$= \mathrm{E}\Big[\left(\frac{\mathrm{d}}{\mathrm{d}t}G(\Phi^v_{-t}(\mathbf{X}^D))\right)F(\mathbf{X}^D)\left(\prod_{k=1}^{\mathbf{X}(D)}\frac{\rho(\phi^v_{-t}(X_k))}{\rho(X_k)}\,\mathrm{Jac}^{\phi^v_{-t}}(X_k)\right) \tag{B.3.11}$$

$$\frac{\det J[D](\phi^v_{-t}(X_1),\ldots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\ldots,X_{\mathbf{X}(D)})}\Big]$$

$$+\,\mathrm{E}\Big[\left(\frac{\mathrm{d}}{\mathrm{d}t}\prod_{k=1}^{\mathbf{X}(D)}\frac{\rho(\phi^v_{-t}(X_k))}{\rho(X_k)}\,\mathrm{Jac}^{\phi^v_{-t}}(X_k)\right)F(\mathbf{X}^D)G(\Phi^v_{-t}(\mathbf{X}^D)) \tag{B.3.12}$$

$$\frac{\det J[D](\phi^v_{-t}(X_1),\ldots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\ldots,X_{\mathbf{X}(D)})}\Big]$$

$$+\,\mathrm{E}\Big[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\det J[D](\phi^v_{-t}(X_1),\ldots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\ldots,X_{\mathbf{X}(D)})}\right)F(\mathbf{X}^D)G(\Phi^v_{-t}(\mathbf{X}^D)) \tag{B.3.13}$$

$$\prod_{k=1}^{\mathbf{X}(D)}\frac{\rho(\phi^v_{-t}(X_k))}{\rho(X_k)}\,\mathrm{Jac}^{\phi^v_t}(X_k)\Big].$$

The claimed integration by parts formula follows by evaluating the above relation at $t=0$. In particular, we use (B.3.8) to evaluate the second term inside the expectation in the right-hand side of the above equality, and we use the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\det J[D](\phi^v_{-t}(X_1),\ldots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\ldots,X_{\mathbf{X}(D)})}$$

$$= -\sum_{i=1}^{\mathbf{X}(D)}\frac{\nabla_{x_i}\det J[D](\phi^v_{-t}(X_1),\ldots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\ldots,X_{\mathbf{X}(D)})}\cdot v(\phi^v_{-t}(X_i)) \tag{B.3.14}$$

to evaluate the third term inside the expectation in the right-hand side of the above equality. Equality (B.3.14) holds P-a.e. which is enough for our purposes. Using the definition of functionals in $\mathcal{S}_D$, one checks that the r.v.

$$G(\mathbf{X}^D)\frac{\mathrm{d}}{\mathrm{d}t}F(\Phi^v_t(\mathbf{X}^D))$$

is uniformly bounded in $t\in I_0$ by a positive constant. By the assumptions (**H2**) and (**H3**) and the form (5.5) of the functionals in $\mathcal{S}_D$, one may easily check that all the terms inside the expectations in the right-hand side of the above equality can be uniformly bounded in $t\in I_0$ by integrable r.v.'s and this justifies the exchange of expectation and derivative in (B.3.9). We check this fact only for the latter term. Take

$$F(\mathbf{X}^D) = f_0\mathbf{1}_{\{\mathbf{X}(D)=0\}} + \sum_{k=1}^{n}\mathbf{1}_{\{\mathbf{X}(D)=k\}}f_k(X_1,\ldots,X_k)$$

of the form (5.5). By (B.3.14) we easily see that the modulus of the r.v.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\det J[D](\phi^v_{-t}(X_1),\ldots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\ldots,X_{\mathbf{X}(D)})}\right)F(\mathbf{X}^D)G(\Phi^v_{-t}(\mathbf{X}^D))\prod_{k=1}^{\mathbf{X}(D)}\frac{\rho(\phi^v_{-t}(X_k))}{\rho(X_k)}\,\mathrm{Jac}^{\phi^v_{-t}}(X_k)$$

up to a positive constant is bounded above P-a.e. by

$$\sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \left(\prod_{i=1}^{k} \frac{\rho(\phi^v_{-t}(X_i))}{\rho(X_i)} \operatorname{Jac}^{\phi^v_{-t}}(X_i) \right) \sum_{i=1}^{k} \left| \frac{\nabla_{x_i} \det J[D](\phi^v_{-t}(X_1), \ldots, \phi^v_{-t}(X_k))}{\det J[D](X_1, \ldots, X_k)} \cdot v(\phi^v_{-t}(X_i)) \right|. \quad \text{(B.3.15)}$$

Since ρ is continuous on D, for any $n \geq 1$, the map

$$(x_1, \ldots, x_n) \longmapsto \det J[D](x_1, \ldots, x_n)$$

is continuously differentiable on D^n, and the Jacobian is given by (B.3.7), up to a positive constant. The term in (B.3.15) is bounded above P-a.e., uniformly in $t \in I_0$, by

$$\sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \frac{\rho(X_1)^{-1} \cdots \rho(X_k)^{-1}}{\det J[D](X_1, \ldots, X_k)}.$$

To conclude the proof, we only need to check that the mean of this r.v. is finite. We have by definition of the Janossy densities, and since $\det J[D](\mathbf{x}) > 0$, for P_D-a.e. $\mathbf{x} \in N_f^D$:

$$\begin{aligned} &\mathrm{E}\left[\mathbf{1}_{\{\mathbf{X}(D)=k\}} \frac{\rho(X_1)^{-1} \cdots \rho(X_k)^{-1}}{\det J[D](X_1, \ldots, X_k)} \right] \quad \text{(B.3.16)} \\ &= \frac{1}{k!} \int_{D^k} \frac{j_D^k(x_1, \ldots, x_k)}{\det J[D](x_1, \ldots, x_k)} \mathbf{1}_{\{j_D^k(x_1, \ldots, x_k) > 0\}} \frac{1}{\rho(x_1)} \lambda(\mathrm{d}x_1) \ldots \frac{1}{\rho(x_k)} \lambda(\mathrm{d}x_k) \\ &= \frac{1}{k!} \int_{D^k} \frac{j_D^k(x_1, \ldots, x_k)}{\det J[D](x_1, \ldots, x_k)} \mathbf{1}_{\{j_D^k(x_1, \ldots, x_k) > 0\}} \ell(\mathrm{d}x_1) \ldots \ell(\mathrm{d}x_k) \\ &= \frac{\operatorname{Det}(\mathbf{Id} - \mathcal{K}_D)}{k!} \ell(D^k) < \infty, \end{aligned}$$

where ℓ denotes the Lebesgue measure, and we have used the fact that $\rho = \frac{\mathrm{d}\lambda}{\mathrm{d}\ell}$. □

Remark B.3.3. *We remark that there is a sign change in (B.3.6), as compared to the results of [14], which is justified by the corrected formula for (5.8). This corrected version is also more in line with the corresponding integration by parts for the Poisson point process.*

Next, we extend the integration by parts formula by closability to a larger class of functionals. For $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$, we consider the closability of the linear operators $\nabla_v^{\mathrm{N}_{lf}} : \mathcal{S}_D \longrightarrow \mathrm{L}^2_D$ and $\nabla_v^{\mathrm{N}_{lf}*} : \mathcal{S}_D \longrightarrow \mathrm{L}^2_D$ defined, respectively, by $F(\mathbf{X}^D) \mapsto \nabla_v^{\mathrm{N}_{lf}} F(\mathbf{X}^D)$ and $F(\mathbf{X}^D) \mapsto \nabla_v^{\mathrm{N}_{lf}*} F(\mathbf{X}^D)$. In addition we state our extension of the integration by parts formula (B.3.6) by closability. In the following, we denote by $\overline{A}$ the minimal closed extension of a closable linear operator A, and by $\mathrm{Dom}(\overline{A})$ the domain of $\overline{A}$.

Theorem B.3.4. *Assume that* (**H1**), (**H2**) *and* (**H3**) *hold, and that*

$$\int_{D^n} \left| \frac{\partial_{x_i^{(h)}} \det J[D](x_1, \ldots, x_n) \partial_{x_j^{(k)}} \det J[D](x_1, \ldots, x_n)}{\det J[D](x_1, \ldots, x_n)} \right| \mathbf{1}_{\{\det J[D](x_1, \ldots, x_n) > 0\}} \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_n) < \infty \quad \text{(B.3.17)}$$

for any $n \geq 1$, $1 \leq i, j \leq n$ and $1 \leq h, k \leq d$. Then

(*i*) *the linear operators* $\nabla_v^{\mathrm{N}_{lf}}$ *and* $\nabla_v^{\mathrm{N}_{lf}*}$ *are well-defined and closable for any vector field* $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$. *In particular, we have*

$$\nabla_v^{\mathrm{N}_{lf}}(\mathcal{S}_D) \subset \mathrm{L}_D^2 \qquad \text{and} \qquad \nabla_v^{\mathrm{N}_{lf}*}(\mathcal{S}_D) \subset \mathrm{L}_D^2;$$

(*ii*) *for any vector field* $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$, *we have*

$$\mathrm{E}\left[\mathbf{G}\overline{\nabla_v^{\mathrm{N}_{lf}}}\mathbf{F}\right] = \mathrm{E}\left[\mathbf{F}\overline{\nabla_v^{\mathrm{N}_{lf}*}}\mathbf{G}\right]$$

for all $\mathbf{F} \in \mathrm{Dom}\left(\overline{\nabla_v^{\mathrm{N}_{lf}}}\right)$, $\mathbf{G} \in \mathrm{Dom}\left(\overline{\nabla_v^{\mathrm{N}_{lf}*}}\right)$ *in the domains of the minimal closed extensions of* $\nabla_v^{\mathrm{N}_{lf}}$ *and* $\nabla_v^{\mathrm{N}_{lf}*}$.

Note that under the assumptions (**H1**), (**H2**) and (**H3**), condition (B.3.17) is satisfied if, for any $n \geq 1$, the function

$$(x_1, \ldots, x_n) \longmapsto \det J[D](x_1, \ldots, x_n)$$

is strictly positive on the compact D^n.

Proof of Theorem 5.3.2. (*i*) Let $v \in \mathcal{C}_c^\infty(D, \mathbb{R}^d)$ and $F(\mathbf{X}^D) \in \mathcal{S}_D$. For ease of notation, throughout this proof we write ∇_v in place of $\nabla_v^{\mathrm{N}_{lf}}$ and ∇_v^* in place of $\nabla_v^{\mathrm{N}_{lf}*}$. We clearly have

$$|\nabla_v F(\mathbf{X}^D)| \leq C$$

for some constant $C > 0$, almost surely, and therefore $\nabla_v(\mathcal{S}_D) \subset \mathrm{L}_D^2$. The claim $\nabla_v^*(\mathcal{S}_D) \subset \mathrm{L}_D^2$ follows if we check that $\|G(\mathbf{X}^D)\nabla_v U[D](\mathbf{X}^D)\|_{\mathrm{L}_D^2} < \infty$ and $\|G(\mathbf{X}^D)B_v^\mu(\mathbf{X}^D)\|_{\mathrm{L}_D^2} < \infty$ for any $G(\mathbf{X}^D) \in \mathcal{S}_D$. The latter relation easily follows noticing that

$$|G(\mathbf{X}^D)B_v^\mu(\mathbf{X}^D)| \leq C$$

for some constant $C > 0$, almost surely. Taking

$$G(\mathbf{X}^D) = g_0 \mathbf{1}_{\{\mathbf{X}(D)=0\}} + \sum_{k=1}^m \mathbf{1}_{\{\mathbf{X}(D)=k\}} g_k(X_1, \ldots, X_k)$$

of the form (5.5), by (B.3.5) we have

$$G(\mathbf{X}^D)\nabla_v U[D](\mathbf{X}^D) = -\sum_{k=1}^m \mathbf{1}_{\{\mathbf{X}(D)=k\}} g_k(X_1, \ldots, X_k) \sum_{i=1}^k \frac{\nabla_{x_i} \det J[D](X_1, \ldots, X_k)}{\det J[D](X_1, \ldots, X_k)} \cdot v(X_i),$$

and for some positive constant $C > 0$,

$$\|G(\mathbf{X}^D)\nabla_v U[D](\mathbf{X}^D)\|_{\mathrm{L}_D^2}^2 = \sum_{k=1}^m \frac{1}{k!} \int_{D^k} g_k^2(x_1, \ldots, x_k) \mathbf{1}_{\{\det J[D](x_1, \ldots, x_k) > 0\}}$$

$$\left(\sum_{i=1}^{k}\frac{\nabla_{x_i}\det J[D](x_1,\dots,x_k)}{\det J[D](x_1,\dots,x_k)}\cdot v(x_i)\right)^2 j_D^k(x_1,\dots,x_k)\mu(dx_1)\cdots\mu(dx_k)$$

$$= \operatorname{Det}(\mathbf{Id}-\mathcal{K}_D)\sum_{k=1}^{m}\frac{1}{k!}\int_{D^k}\frac{g_k^2(x_1,\dots,x_k)}{\det J[D](x_1,\dots,x_k)}$$

$$\mathbf{1}_{\{\det J[D](x_1,\dots,x_k)>0\}}\left(\sum_{i=1}^{k}\nabla_{x_i}\det J[D](x_1,\dots,x_k)\cdot v(x_i)\right)^2\mu(dx_1)\cdots\mu(dx_k)$$

$$\leq C\operatorname{Det}(\mathbf{Id}-\mathcal{K}_D)\sum_{k=1}^{m}\frac{1}{k!}\sum_{1\leq i,j\leq k}\int_{D^k}\mathbf{1}_{\{\det J[D](x_1,\dots,x_k)>0\}}$$

$$\frac{\nabla_{x_i}\det J[D](x_1,\dots,x_k)\cdot v(x_i)\,\nabla_{x_j}\det J[D](x_1,\dots,x_k)\cdot v(x_j)}{\det J[D](x_1,\dots,x_k)}\mu(dx_1)\cdots\mu(dx_k)$$

$$< \infty,$$

where the latter integral is finite by assumption (5.13).

To conclude, we only need to show that ∇_v is closable (the closability of ∇_v^* can be proved similarly). Let $(F_n(\mathbf{X}^D))_{n\geq 1}$ be a sequence in $\mathcal{S}_D$ converging to 0 in L^2_D and such that $\nabla_v F_n(\mathbf{X}^D)$ converges to V in L^2_D as n goes to infinity. We need to show that $V=0$ a.s. We have

$$|\mathrm{E}[G(\mathbf{X}^D)V]| = \lim_{n\to\infty}|\mathrm{E}[G(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)]| = \lim_{n\to\infty}|\mathrm{E}[F_n(\mathbf{X}^D)\nabla_v^* G(\mathbf{X}^D)]| \qquad \text{(B.3.18)}$$
$$\leq \|\nabla_v^* G(\mathbf{X}^D)\|_{\mathrm{L}^2_D}\lim_{n\to\infty}\|F_n(\mathbf{X}^D)\|_{\mathrm{L}^2_D} = 0, \qquad G\in\mathcal{S}_D.$$

Here, the second equality in (B.3.18) follows by the integration by parts formula (B.3.6) of Lemma B.3.2. The fact that $\mathrm{E}[G(\mathbf{X}^D)V]=0$ for all $G(\mathbf{X}^D)\in\mathcal{S}_D$ implies $V=0$ a.s. is a consequence of the density Lemma 5.2.1.

(ii) By *(i)*, both operators ∇_v and ∇_v^* are closable. Take $\mathbf{F}\in\mathrm{Dom}(\overline{\nabla}_v)$, $\mathbf{G}\in\mathrm{Dom}(\overline{\nabla_v^*})$ and let $(F_n(\mathbf{X}^D))_{n\geq1}$, $(G_n(\mathbf{X}^D))_{n\geq1}$ be sequences in $\mathcal{S}_D$ such that $F_n(\mathbf{X}^D)$ converges to $\mathbf{F}$, $G_n(\mathbf{X}^D)$ converges to $\mathbf{G}$, $\nabla_v F_n(\mathbf{X}^D)$ converges to $\overline{\nabla_v}\mathbf{F}$ and $\nabla_v^* G_n(\mathbf{X}^D)$ converges to $\overline{\nabla_v^*}\mathbf{G}$ in L^2_D as n goes to infinity. By Lemma B.3.2 the integration by parts formula applies to r.v.'s in $\mathcal{S}_D$, therefore we have $\mathrm{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)]=\mathrm{E}[F_n(\mathbf{X}^D)\nabla_v^* G_n(\mathbf{X}^D)]$ for all $n\geq1$. The claim follows if we prove that

$$\lim_{n\to\infty}\mathrm{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] = \mathrm{E}[\mathbf{G}\overline{\nabla_v}\mathbf{F}]$$

and

$$\lim_{n\to\infty}\mathrm{E}[F_n(\mathbf{X}^D)\nabla_v^* G_n(\mathbf{X}^D)] = \mathrm{E}[\mathbf{F}\overline{\nabla_v^*}\mathbf{G}].$$

We only show the first limit above; the second limit being proved similarly. We have

$$\begin{aligned}
&|\mathrm{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] - \mathrm{E}[\mathbf{G}\overline{\nabla_v}\mathbf{F}]|\\
&= |\mathrm{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] - \mathrm{E}[G_n(\mathbf{X}^D)\overline{\nabla_v}\mathbf{F}] + \mathrm{E}[G_n(\mathbf{X}^D)\overline{\nabla_v}\mathbf{F}] - \mathrm{E}[\mathbf{G}\overline{\nabla_v}\mathbf{F}]|\\
&\leq |\mathrm{E}[G_n(\mathbf{X}^D)(\nabla_v F_n(\mathbf{X}^D) - \overline{\nabla_v}\mathbf{F})]| + |\mathrm{E}[(G_n(\mathbf{X}^D)-\mathbf{G})\overline{\nabla_v}\mathbf{F}]|\\
&\leq \|G_n(\mathbf{X}^D)\|_{\mathrm{L}^2_D}\|\nabla_v F_n(\mathbf{X}^D) - \overline{\nabla_v}\mathbf{F}\|_{\mathrm{L}^2_D} + \|G_n(\mathbf{X}^D)-\mathbf{G}\|_{\mathrm{L}^2_D}\|\overline{\nabla_v}\mathbf{F}\|_{\mathrm{L}^2_D},
\end{aligned}$$

which tends to 0 as n goes to infinity. □

B.4 Dirichlet forms

In this section we construct the symmetric Dirichlet form associated to a determinantal process (see Theorem B.4.1 below.)

We start by recalling some definitions related to bilinear forms (see [48] for details). Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \times \mathrm{Dom}(\mathcal{A}) \longrightarrow \mathbb{R}$ a bilinear form defined on a dense subspace $\mathrm{Dom}(\mathcal{A})$ of H, the domain of $\mathcal{A}$. The form $\mathcal{A}$ is said to be symmetric if $\mathcal{A}(x, y) = \mathcal{A}(y, x)$, for any $x, y \in \mathrm{Dom}(\mathcal{A})$, and non-negative definite if $\mathcal{A}(x, x) \geq 0$, for any $x \in \mathrm{Dom}(\mathcal{A})$. Let $\mathcal{A}$ be symmetric and non-negative definite, $\mathcal{A}$ is said closed if $\mathrm{Dom}(\mathcal{A})$ equipped with the norm

$$\|x\|_{\mathcal{A}} := \sqrt{\mathcal{A}(x, x) + \langle x, x \rangle}, \quad x \in \mathrm{Dom}(\mathcal{A}),$$

is a Hilbert space. A symmetric and non-negative definite bilinear form $\mathcal{A}$ is said closable if, for any sequence $(x_n)_{n\geq 1} \subset \mathrm{Dom}(\mathcal{A})$ such that x_n goes to 0 in H and $(x_n)_{n\geq 1}$ is Cauchy *w.r.t.* $\| \cdot \|_{\mathcal{A}}$ it holds that $\mathcal{A}(x_n, x_n)$ converges to 0 in $\mathbb{R}$ as n goes to infinity. Let $\mathcal{A}$ be closable and denote by $\mathrm{Dom}(\overline{\mathcal{A}})$ the completion of $\mathrm{Dom}(\mathcal{A})$ *w.r.t.* the norm $\| \cdot \|_{\mathcal{A}}$. It turns out that $\mathcal{A}$ is uniquely extended to $\mathrm{Dom}(\overline{\mathcal{A}})$ by the closed, symmetric and non-negative definite bilinear form

$$\overline{\mathcal{A}}(x, y) = \lim_{n\to\infty} \mathcal{A}(x_n, y_n), \quad (x, y) \in \mathrm{Dom}(\overline{\mathcal{A}}) \times \mathrm{Dom}(\overline{\mathcal{A}}),$$

where $\{(x_n, y_n)\}_{n\geq 1}$ is any sequence in $\mathrm{Dom}(\mathcal{A}) \times \mathrm{Dom}(\mathcal{A})$ such that such that (x_n, y_n) converges to $(x, y) \in \mathrm{Dom}(\overline{\mathcal{A}}) \times \mathrm{Dom}(\overline{\mathcal{A}})$ *w.r.t.* the norm $\| \cdot \|_{\overline{\mathcal{A}}} + \| \cdot \|_{\overline{\mathcal{A}}}$. A symmetric, non-negative definite and closed bilinear form $\mathcal{A}$ is said symmetric coercive closed form if the following weak sector condition is satisfied:

$$\exists \text{ a constant } c > 0 \text{ such that } |\mathcal{A}_1(x, y)| \leq c\, \mathcal{A}_1(x, x)^{1/2} \mathcal{A}_1(y, y)^{1/2}, \qquad x, y \in \mathrm{Dom}(\mathcal{A}), \tag{B.4.1}$$

where

$$\mathcal{A}_1(x, y) := \mathcal{A}(x, y) + \langle x, y \rangle.$$

Suppose $H = \mathrm{L}^2(B, \mathcal{B}, \beta)$ where $(B, \mathcal{B}, \beta)$ is a measure space. In such a case a symmetric coercive closed form $\mathcal{A}$ is said to be a symmetric Dirichlet form if

$$\mathcal{A}(f^+ \wedge 1, f^+ \wedge 1) \leq \mathcal{A}(f, f), \mathcal{A}(f^+ \wedge 1, f^+ \wedge 1) \leq \mathcal{A}(f, f), \qquad f \in \mathrm{Dom}(\mathcal{A}),$$

where f^+ denotes the positive part of f. Suppose that B is a Hausdorff topological space and let $\mathcal{A}$ be a symmetric Dirichlet form. An $\mathcal{A}$-nest is an increasing sequence $(C_n)_{n\geq 1}$ of closed subsets of B such that

$$\bigcup_{n\geq 1} \{f \in \mathrm{Dom}(\mathcal{A}) \colon f = 0 \ \beta\text{-a.e. on } B \setminus C_n\}$$

is dense in $\mathrm{Dom}(\mathcal{A})$ *w.r.t.* the norm $\| \cdot \|_{\mathcal{A}}$. We say that a subset $B' \subset B$ is $\mathcal{A}$-exceptional if there exists an $\mathcal{A}$-nest $(C_n)_{n\geq 1}$ with $B' \subset B \setminus \bigcup_{n\geq 1} C_n$. Throughout this paper we say that a property holds $\mathcal{A}$-almost everywhere ($\mathcal{A}$-a.e.) if it holds up to an $\mathcal{A}$-exceptional set. Moreover, a function $f : B \to \mathbb{R}$ is called $\mathcal{A}$-almost continuous ($\mathcal{A}$-a.c.) if there exists an $\mathcal{A}$-nest $(C_n)_{n\geq 1}$ such that the restriction $f_{|C_n}$ of f to C_n is continuous for each $n \geq 1$.

Let B again a Hausdorff topological space. A symmetric Dirichlet form $\mathcal{A}$ on the Hilbert space $\mathrm{L}^2(B, \mathcal{B}(B), \beta)$ is called quasi-regular if:

(i) There exists an $\mathcal{A}$-nest $(C_n)_{n\geq 1}$ consisting of compact sets.
(ii) There exists a $\|\cdot\|_{\mathcal{A}}$-dense subset of $\mathrm{Dom}(\mathcal{A})$ whose elements have $\mathcal{A}$-a.c. β-versions.
(iii) There exist $f_k \in \mathrm{Dom}(\mathcal{A})$, $k \geq 1$, having $\mathcal{A}$-a.c. β-versions $\tilde{f}_k$, $k \geq 1$, such that $(\tilde{f}_k)_{k\geq 1}$ is a separating set for $B \setminus N$ (i.e. for any $x, y \in B \setminus N$, $x \neq y$, there exists $\tilde{f}_{k^*}$ such that $\tilde{f}_{k^*}(x) \neq \tilde{f}_{k^*}(y)$), where N is a subset of B which is $\mathcal{A}$-exceptional.

We denote by $\ddot{\mathrm{N}}_f^D$ the set of non-negative integer-valued finite measures on D, equipped with the vague topology (recall that this topology is metrized by the so-called "d-hash" metric, see e.g. Appendix A2 pages 402-405 in [17]). For technical reasons, in this section and Section B.5 we shall see N_f^D as a subspace of $\ddot{\mathrm{N}}_f^D$, via the identification

$$\mathbf{x} \equiv \sum_{x\in\mathbf{x}} \delta_x, \qquad \mathbf{x} \in \mathrm{N}_f^D,$$

where δ_x denotes the Dirac measure at $x \in D$. We shall denote by $\mathcal{B}(\ddot{\mathrm{N}}_f^D)$ the corresponding Borel σ-field. Letting $\mathbf{X}$ denote a determinantal point process with kernel K, using obvious notation, we shall identify L^2_D with $\mathrm{L}^2(\ddot{\mathrm{N}}_f^D, P_D)$.

We consider the bilinear map $\mathcal{E}$ defined on $\mathcal{S}_D \times \mathcal{S}_D$ by

$$\mathcal{E}(F(\mathbf{X}^D), G(\mathbf{X}^D)) := \mathrm{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D) \cdot \nabla_{X_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right].$$

For $F(\mathbf{X}^D) \in \mathcal{S}_D$ of the form (5.5), i.e.

$$F(\mathbf{X}^D) = f_0 \mathbf{1}_{\{\mathbf{X}(D)=0\}} + \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} f_k(X_1, \ldots, X_k),$$

we also define the linear Laplace operator $\mathcal{H}$ by

$$\begin{aligned}\mathcal{H}F(\mathbf{X}^D) = \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} \Big(&- \beta^{\mu}(X_i) \cdot \nabla_{x_i} f_k(X_1, \ldots, X_k) \\ &- \Delta_{x_i} f_k(X_1, \ldots, X_k) + U_{i,k}(X_1, \ldots, X_k) \cdot \nabla_{x_i} f_k(X_1, \ldots, X_k)\Big),\end{aligned}$$

where $\Delta = -\mathrm{div}\nabla$ denotes the Laplacian.

In the following, we consider the subspace $\tilde{\mathcal{S}}_D$ of $\mathcal{S}_D$ made of r.v.'s $F(\mathbf{X}^D) \in \mathcal{S}_D$ of the form

$$F(\mathbf{X}^D) = f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_M(X_k)\right) \mathbf{1}_{\{\mathbf{X}(D)\leq K\}},$$

for some integers $M, K \geq 1$, $\varphi_1, \ldots, \varphi_M \in \mathcal{C}^\infty(D)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^M)$. Note that $\tilde{\mathcal{S}}_D$ is dense in L^2_D (see e.g. [48] p. 54).

The next Theorem B.4.1 provides the Dirichlet form associated to a determinantal process.

Theorem B.4.1. *Under the assumptions of Theorem 5.3.2, we have:*
(i) The linear operator $\mathcal{H} : \tilde{\mathcal{S}}_D \longrightarrow \mathrm{L}^2_D$ is symmetric, non-negative definite and well-defined, i.e. $\mathcal{H}(\tilde{\mathcal{S}}_D) \subset \mathrm{L}^2_D$. In particular the operator square root $\mathcal{H}^{1/2}$ of $\mathcal{H}$ exists.

(ii) The bilinear form $\mathcal{E} : \tilde{\mathcal{S}}_D \times \tilde{\mathcal{S}}_D \longrightarrow \mathbb{R}$ *is symmetric, non-negative definite and well-defined, i.e.* $\mathcal{E}(\tilde{\mathcal{S}}_D \times \tilde{\mathcal{S}}_D) \subset \mathbb{R}$.
(iii) $\mathcal{H}^{1/2}$ *and* $\mathcal{E}$ *are closable and the following relation holds:*

$$\overline{\mathcal{E}}(\mathbf{F}, \mathbf{G}) = \mathrm{E}[\overline{\mathcal{H}^{1/2}}\,\mathbf{F}\overline{\mathcal{H}^{1/2}}\,\mathbf{G}], \quad \forall\ \mathbf{F}, \mathbf{G} \in \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}). \tag{B.4.2}$$

(iv) The bilinear form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ *is a symmetric Dirichlet form.*

Proof. The proof of this theorem is based on Lemma B.4.2 below.
(i) By Relation (B.4.4) in Lemma B.4.2 we easily deduce that, for any $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ we have

$$\mathrm{E}[G(\mathbf{X}^D)\mathcal{H}F(\mathbf{X}^D)] = \mathrm{E}[F(\mathbf{X}^D)\mathcal{H}G(\mathbf{X}^D)] \quad \text{and} \quad \mathrm{E}[F(\mathbf{X}^D)\mathcal{H}F(\mathbf{X}^D)] \geq 0.$$

Therefore, $\mathcal{H}$ is symmetric and non-negative definite. It remains to check that, under the foregoing assumptions, $\mathcal{H}$ is well-defined. Let $F(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ be of the form

$$\begin{aligned} F(\mathbf{X}^D) &= \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} h\left(\sum_{i=1}^{k} \varphi_1(X_i), \ldots, \sum_{i=1}^{k} \varphi_m(X_i)\right) \\ &= \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} h_k(X_1, \ldots, X_k) \end{aligned}$$

for some integers $m, n \geq 1$, $\varphi_1, \ldots, \varphi_m \in \mathcal{C}^\infty(D)$, $h \in \mathcal{C}_b^\infty(\mathbb{R}^m)$. By the definition of $\mathcal{H}$, for the well-definiteness of $\mathcal{H}$ we only need to check that $\|\mathbf{1}_{\{\mathbf{X}(D)=k\}} B^{\mu}_{\nabla_{x_i} h_k}(\mathbf{X}^D)\|_{\mathrm{L}^2_D} < \infty$ and $\|\mathbf{1}_{\{\mathbf{X}(D)=k\}} \nabla^{\mathrm{N}_{lf}}_{\nabla_{x_i} h_k} U[D](\mathbf{X}^D)\|_{\mathrm{L}^2_D} < \infty$. The assumptions guarantee these relations, cf. the proof of Theorem 5.3.2.
(ii) The symmetry and non-negative definiteness of $\mathcal{E}$ follow from Lemma B.4.2 below. It remains to check that, under the foregoing assumptions, $\mathcal{E}$ is well-defined. By Step *(i)*, for any $F(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$, we have $\mathcal{H}F(\mathbf{X}^D) \in \mathrm{L}^2_D$. We conclude the proof by noting that, by Lemma B.4.2, for any $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ and some positive constant $c > 0$, we have

$$|\mathcal{E}(F(\mathbf{X}^D), G(\mathbf{X}^D))| = |\mathrm{E}[G(\mathbf{X}^D)\mathcal{H}F(\mathbf{X}^D)]| \leq c\|\mathcal{H}F(\mathbf{X}^D)\|_{\mathrm{L}^2_D} < \infty.$$

(iii) We first show that $\mathcal{E}$ is closable. We apply Lemma 3.4 page 29 in [48]. We start checking that $\mathcal{E}$ satisfies the weak sector condition (B.4.1). By Relation (B.4.5) in Lemma B.4.2 we have, for any $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$,

$$\begin{aligned} |\mathcal{E}_1(F(\mathbf{X}^D), G(\mathbf{X}^D))| &= |\mathrm{E}[\mathcal{H}^{1/2}F(\mathbf{X}^D)\mathcal{H}^{1/2}G(\mathbf{X}^D)] + \mathrm{E}[G(\mathbf{X}^D)F(\mathbf{X}^D)]| \\ &\leq \|\mathcal{H}^{1/2}F(\mathbf{X}^D)\|_{\mathrm{L}^2_D}\|\mathcal{H}^{1/2}G(\mathbf{X}^D)\|_{\mathrm{L}^2_D} + \|F(\mathbf{X}^D)\|_{\mathrm{L}^2_D}\|G(\mathbf{X}^D)\|_{\mathrm{L}^2_D} \\ &\leq 2\mathcal{E}_1(F(\mathbf{X}^D), F(\mathbf{X}^D))^{1/2}\mathcal{E}_1(G(\mathbf{X}^D), G(\mathbf{X}^D))^{1/2}. \end{aligned}$$

It remains to check that if $(F_n(\mathbf{X}^D))_{n\geq 1} \subset \tilde{\mathcal{S}}_D$ is such that $F_n(\mathbf{X}^D)$ converges to 0 in L^2_D, then $\mathcal{E}(G(\mathbf{X}^D), F_n(\mathbf{X}^D))$ converges to 0, for any $G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$. This easily follows by Lemma B.4.2, the Cauchy-Schwarz inequality and the fact that $\mathcal{H}G(\mathbf{X}^D)$ is square integrable (see the proof of Step *(i)*). The closability of $\mathcal{H}^{1/2}$ follows by the closability of $\mathcal{E}$, Relation (B.4.5) in Lemma B.4.2 and Remark 3.2 *(i)* page 29 in [48]. Finally, we prove Relation (B.4.2). Take $\mathbf{F}, \mathbf{G} \in \mathrm{Dom}(\overline{\mathcal{H}^{1/2}})$ and let $(F_n(\mathbf{X}^D))_{n\geq 1}$, $(G_n(\mathbf{X}^D))_{n\geq 1}$ be sequences in $\tilde{\mathcal{S}}_D$ such that $F_n(\mathbf{X}^D)$ converges to $\mathbf{F}$, $G_n(\mathbf{X}^D)$ converges to $\mathbf{G}$, $\mathcal{H}^{1/2}F_n(\mathbf{X}^D)$

converges to $\overline{\mathcal{H}^{1/2}}\,\mathbf{F}$, and $\mathcal{H}^{1/2}G_n(\mathbf{X}^D)$ converges to $\overline{\mathcal{H}^{1/2}}\,\mathbf{G}$ in L^2_D as n goes to infinity. By Lemma B.4.2 we have

$$\mathcal{E}(F_n(\mathbf{X}^D), G_n(\mathbf{X}^D)) = \mathrm{E}[\mathcal{H}^{1/2}F_n(\mathbf{X}^D)\mathcal{H}^{1/2}G_n(\mathbf{X}^D)], \quad \text{for all } n \geq 1.$$

The claim follows if we prove

$$\lim_{n\to\infty} \mathrm{E}[\mathcal{H}^{1/2}F_n(\mathbf{X}^D)\mathcal{H}^{1/2}G_n(\mathbf{X}^D)] = \mathrm{E}[\overline{\mathcal{H}^{1/2}}\,\mathbf{F}\overline{\mathcal{H}^{1/2}}\,\mathbf{G}].$$

We have:

$$\begin{aligned}
&|\mathrm{E}[\mathcal{H}^{1/2}G_n(\mathbf{X}^D)\mathcal{H}^{1/2}F_n(\mathbf{X}^D)] - \mathrm{E}[\overline{\mathcal{H}^{1/2}}\mathbf{G}\,\overline{\mathcal{H}^{1/2}}\,\mathbf{F}]| \\
&= |\mathrm{E}[\mathcal{H}^{1/2}G_n(\mathbf{X}^D)\mathcal{H}^{1/2}F_n(\mathbf{X}^D)] - \mathrm{E}[\mathcal{H}^{1/2}G_n(\mathbf{X}^D)\,\overline{\mathcal{H}^{1/2}}\,\mathbf{F}] \\
&\qquad + \mathrm{E}[\mathcal{H}^{1/2}G_n(\mathbf{X}^D)\,\overline{\mathcal{H}^{1/2}}\,\mathbf{F}] - \mathrm{E}[\overline{\mathcal{H}^{1/2}}\mathbf{G}\,\overline{\mathcal{H}^{1/2}}\,\mathbf{F}]| \\
&\leq |\mathrm{E}[\mathcal{H}^{1/2}G_n(\mathbf{X}^D)(\mathcal{H}^{1/2}F_n(\mathbf{X}^D) - \overline{\mathcal{H}^{1/2}}\,\mathbf{F})]| + |\mathrm{E}[(\mathcal{H}^{1/2}G_n(\mathbf{X}^D) - \overline{\mathcal{H}^{1/2}}\mathbf{G})\overline{\mathcal{H}^{1/2}}\,\mathbf{F}]| \\
&\leq \|\mathcal{H}^{1/2}G_n(\mathbf{X}^D)\|_{\mathrm{L}^2_D}\|\mathcal{H}^{1/2}F_n(\mathbf{X}^D) - \overline{\mathcal{H}^{1/2}}\,\mathbf{F}\|_{\mathrm{L}^2_D} \\
&\qquad + \|\mathcal{H}^{1/2}G_n(\mathbf{X}^D) - \overline{\mathcal{H}^{1/2}}\mathbf{G}\|_{\mathrm{L}^2_D}\|\overline{\mathcal{H}^{1/2}}\,\mathbf{F}\|_{\mathrm{L}^2_D} \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

(iv) The bilinear form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ defined by (B.4.2) is clearly symmetric, non-negative definite, and closed. Using the Cauchy-Schwarz inequality and equality (B.4.2) (i.e. reasoning similarly as the first part of Step (iii)) it is easily checked that the weak sector condition (B.4.1) holds. So by Definition 2.4 page 16 in [48] we have that $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ is a symmetric coercive closed form. We conclude the proof by applying Proposition 4.10 page 35 in [48]. First note that $\tilde{\mathcal{S}}_D$ is dense in $\mathrm{Dom}(\overline{\mathcal{H}^{1/2}})$ (*w.r.t.* the norm $\overline{\mathcal{E}}_1^{1/2}$). By Exercise 2.7 page 47 in [48], for any $\varepsilon > 0$ there exists an infinitely differentiable function $\varphi_\varepsilon : \mathbb{R} \longrightarrow [-\varepsilon, 1+\varepsilon]$ (which has not to be confused with the functions $\varphi_1, \ldots, \varphi_M$ involved in the definition of the r.v. $F(\mathbf{X}^D)$ below) such that $\varphi_\varepsilon(t) = t$ for any $t \in [0,1]$, $0 \leq \varphi_\varepsilon(t) - \varphi_\varepsilon(s) \leq t - s$ for all $t, s \in \mathbb{R}$, $t \geq s$, $\varphi_\varepsilon(t) = 1 + \varepsilon$ for $t \in [1+2\varepsilon, \infty)$ and $\varphi_\varepsilon(t) = -\varepsilon$ for $t \in (-\infty, -2\varepsilon]$. Note that $|\varphi_\varepsilon'(t)|^2 \leq 1$ for any $\varepsilon > 0$, $t \in \mathbb{R}$ and φ_ε is in $\mathcal{C}_b^\infty$, for any $\varepsilon > 0$. Consider the r.v.

$$F(\mathbf{X}^D) = f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_M(X_k)\right)\mathbf{1}_{\{\mathbf{X}(D)\leq K\}},$$

for some integers $M, K \geq 1$, $\varphi_1, \ldots, \varphi_M \in \mathcal{C}^\infty(D)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^M)$. Note that $F(\mathbf{X}^D), \varphi_\varepsilon \circ F(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$. Indeed

$$\begin{aligned}
\varphi_\varepsilon \circ F(\mathbf{X}^D) &= \varphi_\varepsilon\left(f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_M(X_k)\right)\mathbf{1}_{\{\mathbf{X}(D)\leq K\}}\right) \\
&= \varphi_\varepsilon\left(f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_M(X_k)\right)\right)\mathbf{1}_{\{\mathbf{X}(D)\leq K\}},
\end{aligned}$$

because $\varphi_\varepsilon(0) = 0$. By Lemma B.4.2 we have

$$\overline{\mathcal{E}}(\varphi_\varepsilon \circ F(\mathbf{X}^D), \varphi_\varepsilon \circ F(\mathbf{X}^D)) = \mathrm{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}}\varphi_\varepsilon \circ F(\mathbf{X}^D) \cdot \nabla_{X_i}^{\mathrm{N}_{lf}}\varphi_\varepsilon \circ F(\mathbf{X}^D)\right]$$

$$
\begin{aligned}
&= \mathrm{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \|\nabla_{X_i}^{\mathrm{N}_{lf}} \varphi_\varepsilon \circ F(\mathbf{X}^D)\|^2\right] \\
&= \mathrm{E}\left[\sum_{k=1}^{K} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} \|\nabla_{X_i}^{\mathrm{N}_{lf}} \varphi_\varepsilon \circ F(\mathbf{X}^D)\|^2\right] \\
&= \mathrm{E}\left[\sum_{k=1}^{K} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} \Big\| \sum_{m=1}^{M} \varphi_\varepsilon' \circ f\left(\sum_{l=1}^{k} \varphi_1(X_l), \ldots, \sum_{l=1}^{k} \varphi_M(X_l)\right)\right. \\
&\qquad\qquad \left. \times \partial_m f\left(\sum_{l=1}^{k} \varphi_1(X_l), \ldots, \sum_{l=1}^{k} \varphi_M(X_l)\right) \nabla\varphi_m(X_i)\Big\|^2\right] \\
&\le \mathrm{E}\left[\sum_{k=1}^{K} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} \Big\| \sum_{m=1}^{M} \partial_m f\left(\sum_{l=1}^{k} \varphi_1(X_l), \ldots, \sum_{l=1}^{k} \varphi_M(X_l)\right) \nabla\varphi_m(X_i)\Big\|^2\right] \qquad \text{(B.4.3)} \\
&= \overline{\mathcal{E}}(F(\mathbf{X}^D), F(\mathbf{X}^D)),
\end{aligned}
$$

where in (B.4.3) we used the fact that $|\varphi_\varepsilon'(t)|^2 \le 1$, $t \in \mathbb{R}$. By this inequality we easily have, for any $F(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$,

$$
\liminf_{\varepsilon \to 0} \overline{\mathcal{E}}(F(\mathbf{X}^D) \pm \varphi_\varepsilon \circ F(\mathbf{X}^D), F(\mathbf{X}^D) \mp \varphi_\varepsilon \circ F(\mathbf{X}^D)) \ge 0
$$

and the proof is completed (since, as required by Proposition 4.10 page 35 in [48], we checked condition (4.6) page 34 in [48]). Indeed, for any $\varepsilon > 0$, by the above inequality and the symmetry of $\mathcal{E}$ and $\mathcal{H}$, we have

$$
\begin{aligned}
&\overline{\mathcal{E}}(F(\mathbf{X}^D) + \varphi_\varepsilon \circ F(\mathbf{X}^D), F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D)) \\
&= \overline{\mathcal{E}}(F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D), F(\mathbf{X}^D) + \varphi_\varepsilon \circ F(\mathbf{X}^D)) \\
&= \mathrm{E}[(F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D))\mathcal{H}(F(\mathbf{X}^D) + \varphi_\varepsilon \circ F(\mathbf{X}^D))] \\
&= \mathrm{E}[F(\mathbf{X}^D)\mathcal{H}F(\mathbf{X}^D) + F(\mathbf{X}^D)\mathcal{H}\varphi_\varepsilon \circ F(\mathbf{X}^D) \\
&\qquad - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}\varphi_\varepsilon \circ F(\mathbf{X}^D)] \\
&\ge \mathrm{E}[F(\mathbf{X}^D)\mathcal{H}\varphi_\varepsilon \circ F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}F(\mathbf{X}^D)] \\
&= 0.
\end{aligned}
$$

□

Lemma B.4.2. *Under the assumptions of Theorem 5.3.2, for any $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$, we have*

$$
\mathrm{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D) \cdot \nabla_{X_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right] = \mathrm{E}[G(\mathbf{X}^D)\mathcal{H}F(\mathbf{X}^D)] \qquad \text{(B.4.4)}
$$

$$
= \mathrm{E}[\mathcal{H}^{1/2}F(\mathbf{X}^D)\mathcal{H}^{1/2}G(\mathbf{X}^D)]. \qquad \text{(B.4.5)}
$$

Proof. Let $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ be, respectively, of the form

$$
F(\mathbf{X}^D) = f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k)\right) \mathbf{1}_{\{\mathbf{X}(D) \le K_1\}},
$$

$$G(\mathbf{X}^D) = g\left(\sum_{k=1}^{\mathbf{X}(D)} \gamma_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \gamma_{M_2}(X_k)\right) \mathbf{1}_{\{\mathbf{X}(D)\leq K_2\}},$$

for some integers $M_1, M_2, K_1, K_2 \geq 1$, $\varphi_1, \ldots, \varphi_{M_1}, \gamma_1, \ldots, \gamma_{M_2} \in \mathcal{C}^\infty(D)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^{M_1})$, $g \in \mathcal{C}_b^\infty(\mathbb{R}^{M_2})$. Define

$$F_i(\mathbf{X}^D) = \partial_i f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k)\right) \mathbf{1}_{\{\mathbf{X}(D)\leq K_1\}},$$

and

$$v_i(x) = \nabla\varphi_i(x), \quad x \in D.$$

By direct computation we find

$$\begin{aligned}
\mathcal{H}F(\mathbf{X}^D) = {} & -\mathbf{1}_{\{\mathbf{X}(D)\leq K_1\}} \sum_{i=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} \beta^\mu(X_k)\cdot v_i(X_k)\partial_i f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k)\right) \\
& -\mathbf{1}_{\{\mathbf{X}(D)\leq K_1\}} \sum_{i,j=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} v_i(X_k) \sum_{l=1}^{\mathbf{X}(D)} v_j(X_l)\partial_i\partial_j f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k)\right) \\
& +\mathbf{1}_{\{\mathbf{X}(D)\leq K_1\}} \sum_{i=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} \operatorname{div} v_i(X_k)\partial_i f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k)\right) \\
& +\mathbf{1}_{\{\mathbf{X}(D)\leq K_1\}} \sum_{i=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} U_{k,\mathbf{X}(D)}(X_1, \ldots, X_{\mathbf{X}(D)})\cdot v_i(X_k) \\
& \qquad \partial_i f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k)\right) \\
= {} & -\sum_{i=1}^{M_1} F_i(\mathbf{X}^D) \sum_{k=1}^{\mathbf{X}(D)} \beta^\mu(X_k)\cdot v_i(X_k) \\
& -\mathbf{1}_{\{\mathbf{X}(D)\leq K_1\}} \sum_{i,j=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} v_i(X_k) \sum_{l=1}^{\mathbf{X}(D)} v_j(X_l)\partial_i\partial_j f\left(\sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k)\right) \\
& +\sum_{i=1}^{M_1} F_i(\mathbf{X}^D) \sum_{k=1}^{\mathbf{X}(D)} \operatorname{div} v_i(X_k) \\
& +\sum_{i=1}^{M_1} F_i(\mathbf{X}^D)\nabla_{v_i}^{\mathrm{N}_{lf}} U[D](\mathbf{X}^D),
\end{aligned}$$

which yields

$$\begin{aligned}
\mathcal{H}F(\mathbf{X}^D) &= \sum_{i=1}^{M_1} \left(-\nabla_{v_i}^{\mathrm{N}_{lf}} F_i(\mathbf{X}^D) + (B_{v_i}^\mu(\mathbf{X}^D) + \nabla_{v_i}^{\mathrm{N}_{lf}} U[D](\mathbf{X}^D))F_i(\mathbf{X}^D)\right) \\
&= \sum_{i=1}^{M_1} \nabla_{v_i}^{\mathrm{N}_{lf}*} F_i(\mathbf{X}^D).
\end{aligned}$$

So, by Lemma B.3.2 and since $\tilde{\mathcal{S}}_D \subset \mathcal{S}_D$, using obvious notation we have

$$\begin{aligned}
&\mathrm{E}[G(\mathbf{X}^D)\mathcal{H}F(\mathbf{X}^D)] = \sum_{i=1}^{M_1} \mathrm{E}\left[G(\mathbf{X}^D)\nabla_{v_i}^{\mathrm{N}_{lf}*} F_i(\mathbf{X}^D)\right] = \sum_{i=1}^{M_1} \mathrm{E}\left[F_i(\mathbf{X}^D)\nabla_{v_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right] \\
&= \sum_{i=1}^{M_1} \mathrm{E}\left[F_i(\mathbf{X}^D) \sum_{l=1}^{\mathbf{X}(D)} \sum_{j=1}^{M_2} \partial_j g\left(\sum_{m=1}^{\mathbf{X}(D)} \gamma_1(X_m), \ldots, \sum_{m=1}^{\mathbf{X}(D)} \gamma_{M_2}(X_m)\right) \nabla\gamma_j(X_l)\cdot\nabla\varphi_i(X_l)\right] \\
&= \mathrm{E}\left[\sum_{l=1}^{\mathbf{X}(D)} \sum_{i=1}^{M_1} F_i(\mathbf{X}^D)\nabla\varphi_i(X_l) \cdot \sum_{j=1}^{M_2} G_j(\mathbf{X}^D)\nabla\gamma_j(X_l)\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D)\cdot\nabla_{X_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right].
\end{aligned}$$

Here, we have written

$$G_j(\mathbf{X}^D) := \partial_j g\left(\sum_{k=1}^{\mathbf{X}(D)} \gamma_1(X_k), \ldots, \sum_{k=1}^{\mathbf{X}(D)} \gamma_{M_2}(X_k)\right) \mathbf{1}_{\{\mathbf{X}(D)\le K_2\}}.$$

Finally, since $\mathcal{H}$ is symmetric and non-negative definite the square root operator $\mathcal{H}^{1/2}$ is well-defined. Relation (B.4.5) follows by the properties of $\mathcal{H}^{1/2}$. □

We conclude this section with the following remark which provides the semigroup of the Dirichlet form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$. The connection between such semigroup and the transition semigroup of the diffusion associated to the determinantal process will be specified in Theorem B.5.1 below.

Remark B.4.3. *Assume the hypotheses of Theorem 5.3.2. Then* $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ *is a symmetric Dirichlet form. Its generator is by definition the linear operator* $\mathcal{H}_{\mathrm{gen}}\mathbf{F} = \mathbf{G}$*, where* $\mathbf{G}$ *is determined by the domain of the operator*:

$$\begin{aligned}
&\mathrm{Dom}(\mathcal{H}_{\mathrm{gen}}) \\
&:= \left\{\mathbf{F} \in \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}) :\ \exists\, \mathbf{G} \in \mathrm{L}^2_D\, \forall \mathbf{Z} \in \mathrm{Dom}(\overline{\mathcal{H}^{1/2}})\ \overline{\mathcal{E}}(\mathbf{F},\mathbf{Z}) = -\mathrm{E}[\mathbf{G}\mathbf{Z}]\right\}.
\end{aligned}$$

One may easily see that the operator $(-\mathcal{H}_{\mathrm{gen}}, \mathrm{Dom}(\mathcal{H}_{\mathrm{gen}}))$ *is symmetric, non-negative definite and extends* $(\mathcal{H}, \tilde{\mathcal{S}}_D)$*. Moreover*

$$\mathrm{Dom}(\overline{\mathcal{H}^{1/2}}) = \mathrm{Dom}\left((-\mathcal{H}_{\mathrm{gen}})^{1/2}\right)$$

and

$$\overline{\mathcal{E}}(\mathbf{F},\mathbf{G}) = \mathrm{E}[(-\mathcal{H}_{\mathrm{gen}})^{1/2}\,\mathbf{F}(-\mathcal{H}_{\mathrm{gen}})^{1/2}\,\mathbf{G}], \quad \forall\ \mathbf{F},\mathbf{G} \in \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}).$$

By definition, the symmetric semi-group of $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ *is the linear operator* $T_t\mathbf{F} := \exp(t\mathcal{H}_{\mathrm{gen}})\mathbf{F}$*,* $t > 0$*,* $\mathbf{F} \in \mathrm{L}^2_D$ *(which is defined by the spectral theory for symmetric operators on a Hilbert space).*

B.5 Stochastic dynamics and quasi-regularity

In this section we establish the existence of diffusions corresponding to determinantal point processes associated to the Dirichlet form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$, cf. Theorem B.5.1 below.

B.5.1 Associated diffusion process

We start recalling some notions, see Chapters IV and V in [48]. Given π in the set $\mathcal{P}(\ddot{\mathrm{N}}_f^D)$ of the probability measures on $(\ddot{\mathrm{N}}_f^D, \mathcal{B}(\ddot{\mathrm{N}}_f^D))$, we call a π-stochastic process with state space $\ddot{\mathrm{N}}_f^D$ the collection

$$\mathbf{M}_{D,\pi} = (\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbf{M}_t)_{t\geq 0}, (\mathbf{P}_{\mathbf{x}})_{\mathbf{x}\in\ddot{\mathrm{N}}_f^D}, \mathbf{P}_\pi)$$

where $\mathcal{F} := \bigvee_{t\geq 0}\mathcal{F}_t$ is a σ-field on the set $\mathbf{\Omega}$, $(\mathcal{F}_t)_{t\geq 0}$ is the $\mathbf{P}_\pi$-completed filtration generated by the process $\mathbf{M}_t : \mathbf{\Omega} \longrightarrow \ddot{\mathrm{N}}_f^D$ of $(\mathcal{F}, \mathcal{B}(\ddot{\mathrm{N}}_f^D))$-measurable mappings, $\mathbf{P}_{\mathbf{x}}$ is a probability measure on $(\mathbf{\Omega}, \mathcal{F})$ for all $\mathbf{x} \in \ddot{\mathrm{N}}_f^D$, and P_π is the probability measure on $(\mathbf{\Omega}, \mathcal{F})$ defined by

$$\mathbf{P}_\pi(A) := \int_{\ddot{\mathrm{N}}_f^D} \mathbf{P}_{\mathbf{x}}(A)\, \pi(\mathrm{d}\mathbf{x}), \quad A \in \mathcal{F}.$$

A collection $(\mathbf{M}_{D,\pi}, (\theta_t)_{t\geq 0})$ is called a π-time homogeneous Markov process with state space $\ddot{\mathrm{N}}_f^D$ if $\theta_t : \mathbf{\Omega} \longrightarrow \mathbf{\Omega}$ is a shift operator, i.e. $\mathbf{M}_s \circ \theta_t = \mathbf{M}_{s+t}$, $s, t \geq 0$, the map $\mathbf{x} \mapsto \mathbf{P}_{\mathbf{x}}(A)$ is $(\mathcal{B}(\ddot{\mathrm{N}}_f^D), \mathcal{B}(\mathbb{R}))$-measurable for all $A \in \mathcal{F}$, and the time homogeneous Markov property

$$\mathbf{P}_{\mathbf{x}}(\mathbf{M}_t \in A \,|\, \mathcal{F}_s) = \mathbf{P}_{\mathbf{M}_s}(\mathbf{M}_{t-s} \in A), \qquad \mathbf{P}_{\mathbf{x}} - a.s., \quad A \in \mathcal{F}, \quad 0 \leq s \leq t, \quad \mathbf{x} \in \ddot{\mathrm{N}}_f^D$$

holds. Recall that a π-time homogeneous Markov process $(\mathbf{M}_{D,\pi}, (\theta_t)_{t\geq 0})$ with state space $\ddot{\mathrm{N}}_f^D$ is said to be π-tight on $\ddot{\mathrm{N}}_f^D$ if $(\mathbf{M}_t)_{t\geq 0}$ is right-continuous with left limits $\mathbf{P}_\pi$-almost surely; $\mathbf{P}_{\mathbf{x}}(\mathbf{M}_0 = \mathbf{x}) = 1 \;\forall\, \mathbf{x} \in \ddot{\mathrm{N}}_f^D$; the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous; the following strong Markov property holds:

$$\mathbf{P}_{\pi'}(\mathbf{M}_{t+\tau} \in A \,|\, \mathcal{F}_\tau) = \mathbf{P}_{\mathbf{M}_\tau}(\mathbf{M}_t \in A)$$

$\mathbf{P}_{\pi'}$-almost surely for all $\mathcal{F}_t$-stopping time τ, $\pi' \in \mathcal{P}(\ddot{\mathrm{N}}_f^D)$, $A \in \mathcal{F}$ and $t \geq 0$, cf. Theorem IV.1.15 in [48]. In addition, a π-tight process on $\ddot{\mathrm{N}}_f^D$ is said π-special standard process on $\ddot{\mathrm{N}}_f^D$ if for any $\pi' \in \mathcal{P}(\ddot{\mathrm{N}}_f^D)$ which is equivalent to π and all $\mathcal{F}_t$-stopping times τ, $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \tau$ then $\mathbf{M}_{\tau_n}$ converges to $\mathbf{M}_\tau$, $\mathbf{P}_{\pi'}$-almost surely.

The following theorem holds, in which $\mathrm{E}_{\mathbf{x}}$ denotes the expectation under $\mathbf{P}_{\mathbf{x}}$, $\mathbf{x} \in \ddot{\mathrm{N}}_f^D$.

Theorem B.5.1. *Assume the hypotheses of Theorem 5.3.2. Then there exists a P_D-tight special standard process $(\mathbf{M}_{D,P_D}, (\theta_t)_{t\geq 0})$ on $\ddot{\mathrm{N}}_f^D$ with transition semigroup*

$$p_t f(\mathbf{x}) := \mathrm{E}_{\mathbf{x}}[f(\mathbf{M}_t)], \quad \mathbf{x} \in \ddot{\mathrm{N}}_f^D, \quad f : \ddot{\mathrm{N}}_f^D \longrightarrow \mathbb{R} \quad \textit{square integrable.}$$

In addition, $(\mathbf{M}_{D,P_D}, (\theta_t)_{t\geq 0})$ is properly associated with the Dirichlet form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ in the sense that $p_t f$ is an $\overline{\mathcal{E}}$-a.c., P_D-version of $\exp(t\mathcal{H}_{\mathrm{gen}})f$ for all square integrable $f : \ddot{\mathrm{N}}_f^D \longrightarrow \mathbb{R}$ and $t > 0$, and such that

$$\mathbf{P}_{\mathbf{x}}(\{\omega :\ t \mapsto \mathbf{M}_t(\omega) \textit{ is continuous on } [0, +\infty)\}) = 1, \qquad \overline{\mathcal{E}}\textit{-a.e., } \mathbf{x} \in \ddot{\mathrm{N}}_f^D. \tag{B.5.1}$$

Proof. By Lemma B.5.2 below the Dirichlet form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ is quasi-regular, hence by Theorem III.3.5 page 103 in [48], there exists a P_D-tight special standard process on $\ddot{\mathrm{N}}_f^D$, say $(\mathbf{M}_{D,P_D}, (\theta_t)_{t\geq 0})$, whose transition semigroup $(p_t)_{t\geq 0}$ is such that, for any square integrable function $f : \ddot{\mathrm{N}}_f^D \longrightarrow \mathbb{R}$ and $t > 0$, $p_t f$ is a P_D-version of $\exp(t\mathcal{H}_{\mathrm{gen}})f$ and $p_t f$ is $\overline{\mathcal{E}}$-a.c.. Since the form has clearly the local property (see the definition in [48]), by Theorem V.1.5 page 150 in [48] the tight special standard process is a P_D-diffusion associated to the form, i.e. relation (B.5.1) holds for $\overline{\mathcal{E}}$-a.e. $\mathbf{x} \in \ddot{\mathrm{N}}_f^D$. □

For the sake of completeness we remark that by applying Theorem 6.4 page 141 in [48], one has that the P_D-diffusion on $\ddot{\mathrm{N}}_f^D$ properly associated with the Dirichlet form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ (defined in Theorem B.5.1) is unique up to P_D-equivalence. We refer the reader to Definition 6.3 page 140 in [48] for the meaning of this notion.

B.5.2 Quasi-regularity

The following lemma holds.

Lemma B.5.2. *Assume the hypotheses of Theorem 5.3.2. Then the form* $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ *is quasi-regular.*

Proof. As in [88] and [89], we apply Theorem 3.4 of [72]. For this purpose, we have to show that, for some countable dense set $(\widetilde{\mathbf{x}}_i)_{i\geq 1}$ in $\ddot{\mathrm{N}}_f^D$, there exists a countable collection $(F_{ij}(\mathbf{X}^D))_{i,j\geq 1}$ of random variables such that:
(i) For some bounded metric ϱ on $\ddot{\mathrm{N}}_f^D$, which is uniformly equivalent to the "d-hash" metric, we have $\varrho(\mathbf{x}, \widetilde{\mathbf{x}}_i) = \sup_j F_{ij}(\mathbf{x})$, $\overline{\mathcal{E}}$-a.e. $\mathbf{x} \in \ddot{\mathrm{N}}_f^D$, for any $i \geq 1$.
(ii) $F_{ij} : \mathrm{N}_{lf} \longrightarrow \mathbb{R}$ is of the form

$$F_{ij}(\mathbf{x}) = f_{ij}\left(\sum_{x\in\mathbf{x}} h_1(x), \ldots, \sum_{x\in\mathbf{x}} h_N(x)\right), \quad \mathbf{x} \in \mathrm{N}_{lf}$$

for some integer $N \geq 1$, $h_1, \ldots, h_N \in \mathcal{C}_c^\infty(S)$ and f_{ij} weakly differentiable; $(F_{ij}(\mathbf{X}^D))_{i,j\geq 1} \subset \mathrm{Dom}(\overline{\mathcal{H}^{1/2}})$ and, setting $\mathbf{F}_{ij} = F_{ij}(\mathbf{X}^D)$,

$$\overline{\mathcal{E}}(\mathbf{F}_{ij}, \mathbf{F}_{ij}) = \mathrm{E}\left[\sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}(\mathbf{X}^D)\|^2\right], \tag{B.5.2}$$

where $\nabla_x^{\mathrm{N}_{lf}} F_{ij}(\mathbf{x})$ is defined as in (B.3.3) with $n = \infty$ and the symbol ∇_x being the gradient w.r.t. the weak partial derivatives.
(iii) $\sup_{i,j} \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}(\mathbf{X}^D)\|^2 \in \mathrm{L}_D^1$.
Proof of (i). Such a metric ϱ is defined as the restriction on $\ddot{\mathrm{N}}_f^D$ of the following metric on $\ddot{\mathrm{N}}_{lf}$ (which we continue to denote by ϱ) constructed in [89] as follows. Here $\ddot{\mathrm{N}}_{lf}$ denotes the set of non-negative integer-valued locally finite measures on S, equipped with the vague topology. Let $\mathcal{T}_{\mathrm{b}}$ be a countable open base in S and, for every $O \in \mathcal{T}_{\mathrm{b}}$, take a sequence $\ell_{O_1}, \ell_{O_2}, \ldots \in \mathcal{C}_c^\infty(S)$ such that $\ell_{O_n}(x) \uparrow \mathbf{1}_{\{x\in O\}}$, $\forall\, x \in S$. For ease of notation set $\ell_n = \ell_{O_n}$ and assume that this enumeration satisfies

$$\bigcup_{k=1}^{j} \mathrm{supp}(\ell_k) \subseteq \{x \in S :\ \|x\| \leq j\}, \quad j = 1, 2, \ldots.$$

The desired metric is defined by

$$\varrho(\mathbf{x},\mathbf{y}) := \sup_j 2^{-j}\left\{1-\exp\left(-\frac{\left|\sum_{x\in\mathbf{x}}\ell_j(x)-\sum_{y\in\mathbf{y}}\ell_j(y)\right|}{\overline{\ell}_j}\right)\right\}, \quad \mathbf{x},\mathbf{y}\in\ddot{\mathrm{N}}_{lf}$$

where, for $x\in\mathbb{R}^d$,

$$\overline{\ell}_j := \sup\left\{\left|\frac{\partial\ell_j(x)}{\partial x^{(q)}}\right| :\ x\in S,\, q=1,\ldots,d\right\} > 0.$$

Let $(\widetilde{\mathbf{x}}_i)_{i\geq 1}$ be a countable dense set in $\ddot{\mathrm{N}}_f^D$ and define

$$F_{ij}(\mathbf{x}) := 2^{-j}\left\{1-\exp\left(-\frac{\left|\sum_{x\in\mathbf{x}}\ell_j(x)-\sum_{\widetilde{x}_i\in\widetilde{\mathbf{x}}_i}\ell_j(\widetilde{x}_i)\right|}{\overline{\ell}_j}\right)\right\}, \quad \mathbf{x}\in\ddot{\mathrm{N}}_{lf}.$$

Then (i) follows.

Proof of (ii). Clearly F_{ij} has the claimed form. We now check that $\mathbf{F}_{ij}\in\mathrm{Dom}(\overline{\mathcal{H}^{1/2}})$, i.e. there exists a sequence $(F_{ij}^{(n)}(\mathbf{X}^D))_{n\geq 1}\subset\tilde{\mathcal{S}}_D$ such that, as n goes to infinity, $F_{ij}^{(n)}(\mathbf{X}^D)$ converges to $\mathbf{F}_{ij}$ in L^2_D and $(\mathcal{H}^{1/2}F_{ij}^{(n)}(\mathbf{X}^D))_{n\geq 1}$ converges in L^2_D. Define the function

$$\overline{f}_{ij}(x) := \Big|x-\sum_{\widetilde{x}_i\in\widetilde{\mathbf{x}}_i}\ell_j(\widetilde{x}_i)\Big|, \quad x\in\mathbb{R}$$

and consider the classical sequence of mollifiers defined by $\eta_n(x) := Cn\eta(nx)$, $n\geq 1$, where

$$\eta(x) := \mathbf{1}_{\{|x|<1\}}\mathrm{e}^{-1/(1-x^2)}, \quad x\in\mathbb{R}$$

and $C := \left(\int_{-1}^1\eta(x)\,\mathrm{d}x\right)^{-1}$. Define the functions

$$f_{ij}^{(n)}(x) := 2^{-j}\left\{1-\exp\left(-\frac{\eta_n*\overline{f}_{ij}(x)}{\overline{\ell}_j}\right)\right\}, \quad x\in\mathbb{R}$$

where $*$ denotes the convolution product, and set $F_{ij}^{(n)}(\mathbf{x}) := f_{ij}^{(n)}\left(\sum_{x\in\mathbf{x}}\ell_j(x)\right)\mathbf{1}_{\{\sharp(\mathbf{x})\leq n\}}$, $\mathbf{x}\in\mathrm{N}_f^D$, where the symbol $\sharp(\mathbf{x})$ denotes the number of points in the configuration $\mathbf{x}$. Since $\eta_n\in\mathcal{C}_c^\infty(\mathbb{R})$ and $\overline{f}_{ij}$ is locally integrable on $\mathbb{R}$, we have $\eta_n*\overline{f}_{ij}\in\mathcal{C}^\infty(\mathbb{R})$ (see e.g. Proposition IV.20 in [11]). Therefore $f_{ij}^{(n)}\in\mathcal{C}_b^\infty(\mathbb{R})$ and so $F_{ij}^{(n)}(\mathbf{X}^D)\in\tilde{\mathcal{S}}_D$. Since $\overline{f}_{ij}\in\mathcal{C}(\mathbb{R})$, we deduce that $\eta_n*\overline{f}_{ij}$ converges to $\overline{f}_{ij}$ uniformly on the compacts of $\mathbb{R}$ (see e.g. Proposition IV.21 in [11]), and so $F_{ij}^{(n)}(\mathbf{X}^D)$ converges to $\mathbf{F}_{ij}$ in L^2_D, as n goes to infinity. It remains to show that the sequence $(\mathcal{H}^{1/2}F_{ij}^{(n)}(\mathbf{X}^D))_{n\geq 1}$ converges in L^2_D. For this we are going to prove that it is a Cauchy sequence. We start by noting that since η_n is integrable on $\mathbb{R}$ and $\overline{f}_{ij}$ is weakly differentiable we have (see e.g. Lemma VIII.4 in [11])

$$(\eta_n*\overline{f}_{ij})'(x) := \frac{\mathrm{d}}{\mathrm{d}x}\eta_n*\overline{f}_{ij}(x) = \eta_n*\frac{\mathrm{d}}{\mathrm{d}x}\overline{f}_{ij}(x)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{f}_{ij}(x) = \mathbf{1}_{\left\{\sum_{\widetilde{x}_i\in\widetilde{\mathbf{x}}_i}\ell_j(\widetilde{x}_i)\in(x,\infty)\right\}} - \mathbf{1}_{\left\{\sum_{\widetilde{x}_i\in\widetilde{\mathbf{x}}_i}\ell_j(\widetilde{x}_i)\in(-\infty,x)\right\}}.$$

Thus $|(\eta_n * \overline{f}_{ij})'| \leq 1$, for any $n \geq 1$. Moreover, $(\eta_n * \overline{f}_{ij})'(x)$ converges to $\overline{f}'_{ij}(x)$ point-wise, as n goes to infinity. We also note that for any $n \geq 1$ and $x \in D$ we have

$$\begin{aligned}\nabla_x^{\mathrm{N}_{lf}} F_{ij}^{(n)}(\mathbf{X}^D) &= \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{h=1}^{k} \mathbf{1}_{\{X_h\}}(x) \nabla_{x_h} f_{ij}^{(n)} \left(\sum_{q=1}^{k} \ell_j(X_q) \right) \\ &= \mathbf{1}_{\{\mathbf{X}(D)\leq n\}} \sum_{h=1}^{\mathbf{X}(D)} \mathbf{1}_{\{X_h\}}(x) \nabla_{x_h} f_{ij}^{(n)} \left(\sum_{q=1}^{\mathbf{X}(D)} \ell_j(X_q) \right)\end{aligned}$$

where, for any $\mathbf{x} \in \mathrm{N}_f^D$,

$$\nabla_x f_{ij}^{(n)} \left(\sum_{x \in \mathbf{x}} \ell_j(x) \right) = (2^{-j}/\overline{\ell}_j)(\eta_n * \overline{f}_{ij})' \left(\sum_{x \in \mathbf{x}} \ell_j(x) \right) \exp\left(-\eta_n * \overline{f}_{ij} \left(\sum_{x \in \mathbf{x}} \ell_j(x) \right) / \overline{\ell}_j \right) \nabla \ell_j(x).$$

So, for any $n \geq 1$, i, j and $k = 1, \ldots, \mathbf{X}(D)$, we have

$$\|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 \leq d 4^{-j}, \quad \text{a.s..} \tag{B.5.3}$$

For any $n, m \geq 1$, we deduce

$$\begin{aligned}&\|\mathcal{H}^{1/2} F_{ij}^{(n)}(\mathbf{X}^D) - \mathcal{H}^{1/2} F_{ij}^{(m)}(\mathbf{X}^D)\|_{\mathrm{L}_D^2}^2 = \mathrm{E}[|\mathcal{H}^{1/2}(F_{ij}^{(n)}(\mathbf{X}^D) - F_{ij}^{(m)}(\mathbf{X}^D))|^2] \\ &= \mathrm{E}\left[\sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} [F_{ij}^{(n)}(\mathbf{X}^D) - F_{ij}^{(m)}(\mathbf{X}^D)]\|^2 \right] \\ &\leq d\, 4^{-j} \mathrm{E}\Bigg[\Bigg(\mathbf{1}_{\{\mathbf{X}(D)\leq n\}} (\eta_n * \overline{f}_{ij})' \left(\sum_{k=1}^{\mathbf{X}(D)} \ell_j(X_k) \right) \exp\left(-\eta_n * \overline{f}_{ij} \left(\sum_{k=1}^{\mathbf{X}(D)} \ell_j(X_k) \right) / \overline{\ell}_j \right) \\ &\quad - \mathbf{1}_{\{\mathbf{X}(D)\leq m\}} (\eta_m * \overline{f}_{ij})' \left(\sum_{k=1}^{\mathbf{X}(D)} \ell_j(X_k) \right) \exp\left(-\eta_m * \overline{f}_{ij} \left(\sum_{k=1}^{\mathbf{X}(D)} \ell_j(X_k) \right) / \overline{\ell}_j \right) \Bigg)^2 \mathbf{X}(D) \Bigg].\end{aligned}$$

Due to the above remarks, applying Lebesgue's dominated convergence theorem (recall that $\mathbf{X}(D)$ has a finite mean since it has exponential tails) one has that this quantity goes to zero as n, m go to infinity, and the claim follows. It remains to check (B.5.2). By Lemma B.4.2 we have

$$\mathrm{E}\left[\sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 \right] = \mathrm{E}[(\mathcal{H}^{1/2} F_{ij}^{(n)}(\mathbf{X}^D))^2].$$

Clearly

$$\lim_{n\to\infty} \mathrm{E}[(\mathcal{H}^{1/2} F_{ij}^{(n)}(\mathbf{X}^D))^2] = \mathrm{E}[(\overline{\mathcal{H}^{1/2}} \mathbf{F}_{ij})^2] = \overline{\mathcal{E}}(\mathbf{F}_{ij}, \mathbf{F}_{ij}),$$

and so we only need to check

$$\lim_{n\to\infty} \mathrm{E}\left[\sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 \right] = \mathrm{E}\left[\sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}(\mathbf{X}^D)\|^2 \right].$$

This easily follows by Lebesgue's dominated convergence theorem. Indeed, by (B.5.3), for any i, j, n,

$$\sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 \leq 4^{-j} d\, \mathbf{X}(D), \quad \text{a.s.} \tag{B.5.4}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 = \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}(\mathbf{X}^D)\|^2, \quad \text{a.s..}$$

Proof of (*iii*). It suffices to note that by (B.5.4) we have

$$\sup_{i,j} \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathrm{N}_{lf}} F_{ij}(\mathbf{X}^D)\|^2 \leq \frac{d}{4}\, \mathbf{X}(D), \quad \text{a.s..}$$

□

Appendix C

Perfect simulation of determinantal point processes

C.1 Introduction

Determinantal point process stems back from 1975 when O. Maachi introduce it as the 'fermion' process with repulsive feature on its points. It is only in the last two decades that Soshnikov(2000) [80] and Shirai and Takahashi(2003) [78] studied further on the mathematical properties of the process structure. Classical modeling of repulsive point processes are in essence through Gibbs point processes. Since their introduction, DPP have found many applications in random matrix theory and quantum physics.

Simulations of determinantal point processes are mostly based on the idea of Hough et al. in [34] and further discussed in details in [44]. The main drawback of the idea is due to its large rejections in the sampling. On the other hand, in a Markov chain Monte Carlo method, many of the simulation algorithms are based on the long-running constructions of Markov chain that converges to an equilibrium distribution. The difficulties was in determining the number of steps needed to have such a convergence. Thanks to Propp and Wilson [70], which suggest with the application of coupling theory, we are able to 'exactly' simulate a finite state Markov chain with the desired equilibrium distribution. Although perfect simulation is obviously appealing but we share a common drawback with [34] idea: given a DPP kernel with no explicitly known spectral representation, the statistics (Papangelou conditional intensity) involve in the configurations, it is not simple to extract them from. In general numerical techniques such as Fourier expansion are required because of the limitation of analytical results.

This paper serves as the continuation of work on [34] and [44] in simulating DPP; in [40] uses the CFTP perfect simulation approach on spatial point processes, while here we analyze the application of CFTP simulation on determinantal point processes and we provide some results on the lower and upper bounds for the coalescence time in general case.

This paper is organized as follows. We start in Section C.2 by summarizing the basic notation and recall the definition of a point process, including determinantal point process and its Papangelou conditional intensity. Section C.3 is devoted to details of perfect simulation of DPP via dominated coupling of Markov chains, i.e. CFTP. In Section C.4, we discuss the asymptotic behavior of the running time of CFTP simulation. Lastly in Section C.5, we apply CFTP algorithm to three stationary DPP models defined by the commonly

used covariance functions in multivariate statistical analysis, i.e. Gaussian model, Matérn model and the Cauchy model.

The statistical analysis conducted in this paper are with R (`spatstat` library by Braddeley and Turner).

C.2 Preliminaries

For $(E, \mathcal{B}, \nu)$ a measure space endowed with a Polish space E and a Radon measure ν on E. We denote by $\mathcal{X}$ the set of locally finite point configurations in E:

$$\mathcal{X} := \{\xi \subset E : |\xi \cap \Lambda| < \infty, \quad \forall \text{ compact } \Lambda \subseteq E\}$$

equipped with the σ-field

$$\mathcal{F} := \sigma(\{\xi \in \mathcal{X} : |\xi \cap \Lambda| = i\}, \quad i \geq 0, \forall \text{ compact } \Lambda \subseteq E)$$

where $|X|$ denotes the cardinality of a set X.

A point process X on E is then a random point configuration, i.e. a random integer-valued Radon measure on E and $X(\Lambda)$ represents the number of points that fall in Λ. If $X(\{x\}) \in \{0, 1\}$ a.s. for all $x \in E$, then X is called simple.

The joint intensities ρ_k of a simple point process X is the intensity measure of the set of ordered k-tuples of distinct points of X, $X^{\wedge k}$. More precisely, for any family of mutually disjoint subsets $\Lambda_i \subseteq E$, a simple point process X w.r.t. the measure ν, denote $\rho_k : E^k \to [0, \infty)$ for $k \geq 1$:

$$\mathrm{E}\left[\prod_{i=1}^{k} X(\Lambda_i)\right] = \int_{\prod_i \Lambda_i} \rho_k(x_1, \ldots, x_k) d\nu(x_1) \ldots d\nu(x_k).$$

We assume that $\rho_k(x_1, \ldots, x_k)$ vanish if $x_i = x_j$ for some $i \neq j$, see [35]. Let $\mathbb{K}(x, y) : E^2 \to \mathbb{C}$ be a measurable function, locally square integrable on E^2.

Definition 2. *Determinantal (fermion) point process with kernel $\mathbb{K}$ is defined to be a simple point process X on E which satisfies:*

$$\rho_k(x_1, \ldots, x_k) = \det(\mathbb{K}(x_i, x_j))_{1 \leq i,j \leq k},$$

$\forall k \geq 1$ and $x_1, \ldots, x_k \in E$, ρ_k w.r.t the measure ν

Definition 3. *For a function $f \in L^2$ defined on a compact support Λ, an integral operator $\mathcal{K} : L^2(E, \nu) \to L^2(E, \nu)$ corresponding to $\mathbb{K}$ is defined such that:*

$$\mathcal{K}f(x) = \int_E \mathbb{K}(x, y) f(y) d\nu(y), \text{ for a.e. } x \in E$$

and the associated bounded linear operator $\mathcal{K}_\Lambda$ on $L^2(\Lambda, \nu)$ as:

$$\mathcal{K}_\Lambda f(x) = \int_\Lambda \mathbb{K}(x, y) f(y) d\nu(y), \text{ for a.e. } x \in \Lambda.$$

By spectral theorem, for a compact and self-adjoint operator $\mathcal{K}_\Lambda$, there is an orthonormal basis $\{\varphi_i^\Lambda\}_{i\geq 1}$ of eigenfunctions for $\mathcal{K}_\Lambda$ on $L^2(\Lambda,\nu)$. Consequently, the kernel $\mathbb{K}$ has a spectral representation:

$$\mathbb{K}_\Lambda(x,y)=\sum_{i\geq 1}\lambda_i^\Lambda\varphi_i^\Lambda(x)\overline{\varphi_i^\Lambda(y)}\quad \text{for } x,y\in\Lambda \tag{C.2.1}$$

Definition 4. *A bounded linear operator $\mathcal{K}_\Lambda$ is said to be of trace class whenever for a complete orthonormal basis $\{\varphi_i^\Lambda\}_{i\geq 1}$ of $L^2(\Lambda,\nu)$,*

$$\begin{aligned}\mathbb{K}_\Lambda(x,y) &= \sum_{i\geq 1}\lambda_i^\Lambda\varphi_i^\Lambda(x)\overline{\varphi_i^\Lambda(y)}\quad \textit{for } x,y\in\Lambda\\ \textit{and}\qquad \sum_{i\geq 1}|\lambda_i^\Lambda| &< \infty.\end{aligned}$$

Further we define:

$$\mathrm{Tr}\ \mathcal{K}_\Lambda:=\sum_{i\geq 1}\lambda_i^\Lambda$$

Throughout this paper, we assume that the integral operator of a determinantal point process satisfies the following hypothesis:

Hypothesis (H1): $\mathcal{K}$ is self-adjoint and of trace class, and its spectrum is contained in $[0,1[$, i.e. $0\leq\mathcal{K}\leq Id$ in the operator ordering, and $\|\mathcal{K}\|<1$, where Id denotes the identity operator on $L^2(E,\nu)$.

Suppose that a DPP X defined on E with its kernel $\mathbb{K}(x,y)=\sum_{i\geq 1}\lambda_i\varphi_i(x)\overline{\varphi_i(y)}$, then it has a pleasant property that the size of its configuration is an infinite sum of independent Bernoulli random variables with parameters equal to the eigenvalues, see [34]. Consequently we have:

$$\mathrm{E}[|X|]=\sum_{i=1}^{\infty}\lambda_i\qquad \mathrm{Var}[|X|]=\sum_{i=1}^{\infty}\lambda_i(1-\lambda_i) \tag{C.2.2}$$

Definition 5. *For a trace-class operator $\mathcal{K}$, the Fredholm determinant is defined by:*

$$\mathrm{Det}(I-\mathcal{K})=\exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\mathrm{Tr}(\mathcal{K}^n)\right).$$

From [80], given an arbitrary compact set $\Lambda\subseteq E$, the Janossy density $j_\Lambda(\cdot)$ of a DPP configuration ξ in Λ is given by:

$$j_\Lambda(\xi)=\mathrm{Det}(I-\mathcal{K}_\Lambda)\mathrm{det}J_\Lambda(\xi), \tag{C.2.3}$$

where we define $J_\Lambda:\Lambda^2\to\mathbb{C}$ by

$$J_\Lambda(x,y)=\sum_{i\geq 1}\frac{\lambda_i^\Lambda}{1-\lambda_i^\Lambda}\varphi_i^\Lambda(x)\overline{\varphi_i^\Lambda(y)}\quad \text{for } x,y\in\Lambda, \tag{C.2.4}$$

and given a configuration $\xi=\{x_1,x_2,\ldots,x_n\}$, $J_\Lambda(\xi)$ is a $n\times n$ matrix with

$$J_\Lambda(\xi)_{(i,j)}=J_\Lambda(x_i,x_j),\qquad \forall\ 0<i,j\leq n.$$

We define $\mathrm{det}J_\Lambda(\emptyset,\emptyset)=1$. For details on Janossy density, see [17].

In the following, we summarize some facts concerning DPP Papangelou (conditional) intensity which characterize the local dependence of particles. See [18] [63] for details on Papangelou intensity.

Definition 6. *Given a compact set $\Lambda \subseteq E$ and a simple finite point process X defined on E with its Janossy density j_Λ, the Papangelou intensity of X in Λ defined as:*

$$c_\Lambda(\xi, x) = \frac{j_\Lambda(\xi \cup x)}{j_\Lambda(\xi)}, \quad \text{where } \xi \in \mathcal{X},\ x \in \Lambda \backslash \xi.$$

If $j_\Lambda(\xi) = 0$ then $c(\xi, x) = 0$.

For a Poisson point process with intensity function ρ, $c(\xi, x)$ is independent of ξ, i.e. $c(\xi, x) = \rho(x)$. In general, X is characterized as:

$$\begin{aligned} \text{attractive if}: &\quad c(\xi, x) \leq c(\eta, x) \text{ whenever } x \notin \xi \subseteq \eta \\ \text{repulsive if}: &\quad c(\xi, x) \geq c(\eta, x) \text{ whenever } x \notin \xi \subseteq \eta \end{aligned}$$

Definition 7 ([18]). *For any simple point process X defined on E, where $\psi \in \mathcal{B}$, $\zeta \in \mathcal{F}$ and μ is the distribution of X on $\mathcal{X}$, its modified Campbell measure C_μ on the product space $(\mathcal{X} \times E, \mathcal{F} \otimes \mathcal{B})$:*

$$C_\mu(\zeta \times \psi) = \int_\psi \sum_{x \in X} \delta_{\{(X \backslash x, x) \in (\zeta \times \psi)\}}\ \mu(dX),$$

where δ is the Dirac measure.

Now, we make an additional assumption that $C_\mu \ll \nu \otimes \mu$. From the Definition 6 and (C.2.3), the Papangelou intensity of DPP then follow from the following proposition which we borrow from [28].

Proposition C.2.1. *Given $C_\mu \ll \nu \otimes \mu$, $\forall$ compact $\Lambda \subseteq E$, DPP Papangelou intenstity is given by:*

$$c_\Lambda(\xi, x) = \frac{\det J_\Lambda(\xi \cup x)}{\det J_\Lambda(\xi)}, \quad \text{where } \xi \in \mathcal{X} \text{ and } x \in \Lambda \backslash \xi.$$

If $\det J_\Lambda(\xi) = 0$, define $c_\Lambda(\xi, x) = 0$.

Since $J_\Lambda(\cdot, \cdot)$ is a positive semi-definite matrix, which can be written in the form of:

$$J_\Lambda(\xi x, \xi x) = \begin{pmatrix} A_{\xi,\xi} & A_{\xi,x} \\ A_{x,\xi} & A_{x,x} \end{pmatrix}, \quad \text{where } A_{x,\xi} = A_{\xi,x}^\dagger.$$

Suppose that $\det A_{\xi,\xi} \neq 0$, then

$$\begin{pmatrix} A_{\xi,\xi} & A_{\xi,x} \\ A_{x,\xi} & A_{x,x} \end{pmatrix} = \begin{pmatrix} A_{\xi,\xi} & 0 \\ A_{x,\xi} & I \end{pmatrix} \begin{pmatrix} I & (A_{\xi,\xi})^{-1} A_{\xi,x} \\ 0 & A_{x,x} - A_{x,\xi}(A_{\xi,\xi})^{-1} A_{\xi,x} \end{pmatrix}$$

and

$$\det(J_\Lambda(\xi x, \xi x)) = \det(A_{\xi,\xi}) \det(A_{x,x} - A_{x,\xi}(A_{\xi,\xi})^{-1} A_{\xi,x}).$$

From Hypothesis **(H1)**, we know:

$$\det(A_{\xi,x}^\dagger (A_{\xi,\xi})^{-1} A_{\xi,x}) \geq 0$$

and the following Lemma C.2.2 follows.

Lemma C.2.2. *Papangelou intensity of a DPP is upper bounded:*

$$\max_{\xi\in\mathcal{X},x\in\Lambda} c_\Lambda(\xi,x) = \max_{\xi\in\mathcal{X},x\in\Lambda} \frac{\det(J_\Lambda(\xi\cup x))}{\det(A_{\xi,\xi})} \leq \max_{x\in\Lambda} J_\Lambda(x,x), \tag{C.2.5}$$

and for a stationary model DPP:

$$\max_{\xi\in\mathcal{X},x\in\Lambda} c_\Lambda(\xi,x) \leq H.$$

where a stationary model DPP is a DPP with kernel of the form: $\mathbb{K}(x,y) = \mathbb{K}_0(x-y)$. *Thus, its* $J_\Lambda(x,x)$ *is equal to a constant* H, *for all* $x\in\Lambda$.

Remark C.2.3. *From the above lemma, we have also shown that a DPP is a repulsive point process where its Papangelou intensity:* $c(\xi,x)\leq c(\eta,x)$ *for* $x\notin\eta\subseteq\xi$.

C.3 Simulation

C.3.1 Perfect Simulation via Dominating CFTP

In this section, we describe the idea of using spatial birth and death process to couple from the past of a continuous time Markov chain $X := \{X_t : t\geq 0\}$ defined on $\mathcal{X}$. From here, we restrict all the following point processes to be defined on a Polish space E.

A birth and death process $(X_t)_{t\geq 0}$ with birth rate b and death rate d is a homogenous Markovian process defined on $\mathcal{X}$. Birth rate b and death rate d are non-negative functions defined on $\mathcal{X}\times E$. The process X_t is right-continuous and piecewise constant except at jump times $T_1 < T_2 < \ldots$, where we define the parameters:

$$B(\xi) := \int_E b(\xi,x)d\nu(x), \quad \delta(\xi) := \begin{cases} \sum_{x\in\xi} d(\xi\backslash x,x) & \text{if } \xi\neq\emptyset \\ 0 & \text{if } \xi=\emptyset \end{cases}$$

and we have:

$$T_{m+1} - T_m \sim \text{Exponential}(B(\xi)+\delta(\xi))$$

By conditioning on T_{m+1}, a birth occurs with probability:

$$\frac{B(\xi)}{B(\xi)+\delta(\xi)}$$

and a death with probability:

$$\frac{\delta(\xi)}{B(\xi)+\delta(\xi)}.$$

We denote the size of the birth and death process as $(Z_t)_{t\in\mathbb{R}} := (|X_t|)_{t\in\mathbb{R}}$, where Z_t is a $M/M/\infty$ queue defined on $\mathbb{N}\cup\{0\}$. Its arrival rate is defined: $\lambda := \int_E b(\xi,x)d\nu(x)$ and service rate is defined: $\mu := d$. We denote this process as $M/M/\infty$ (λ,μ). As the total service rate of a $M/M/\infty$ queue is proportional to its size, the process is always stable.

A Markov Chain Monte Carlo method is to let the birth rate b equals to the Papangelou intensity of the desired point process X and the death rate $d=1$, then the Markov chain $\hat{X} = \{\hat{X}_t : t\in\mathbb{R}\}$ constructed converges to the distribution of X. To obtain a perfect simulation, Kendall and Møller [41] introduce the dominating process $(D_t)_{t\in\mathbb{R}}$ as a birth and death process with birth rate $b=H$ and death rate $d=1$ defined on $\mathcal{X}$, where H is the upper bound of the Papangelou intensity of the desired point process X. The intuition

is to introduce a coupling (pairs of Markov chains, L_{T_i} and U_{T_i}) from the past, deriving from D_t that respect a partial ordering on the state space X corresponding to $\hat{X}_t$ under time-evolution, i.e.:

$$L_t \subseteq \hat{X}_t \subseteq U_t \subseteq D_t \qquad \forall s \leq t \leq 0, \tag{C.3.1}$$

$$L_t = U_t \quad \text{if} \quad L_s = U_s \qquad \forall s \leq t \leq 0, \tag{C.3.2}$$

Next, we define a marking process $(M_t)_{t\in\mathbb{R}}$ which is independent from D_t and such that we can adaptively contruct $\hat{X}$, L and U as functionals of (D, M) . We denote the process $\hat{X}^n$ as the constructed Markov chain that begun at time $-n < 0$. Suppose that we have coalescence between L^n and U^n during the progression from time $-n$ to 0, then the exact equilibrium is attained by $\hat{X}^n$ at time 0, where we have $L_0^n = \hat{X}_0^n = U_0^n$.

Remark C.3.1. *Technically, we have to progressively increase n if coalescence failed at time 0, however, in [70] suggested that it is efficient to iteratively doubling n, and we let $n \in \{\frac{1}{2}, 1, 2, 4, \cdots\}$. In later section Section C.4, we will show that the running time is bounded in asymptotic sense. Note that, D is extended for each doubling of n, i.e. $D_{[-2n,0]}$ is computed from extension of $D_{[-n,0]}$.*

Following are the configuration of a perfect simulation. For each jump times $\{T_1 < T_2 < \ldots\}$ in the dominating process D:
if there is a birth of point $x \in \Lambda \backslash D_{T_{i-1}}$ at time T_i, we set:

$$\begin{aligned} U_{T_i} &:= U_{T_{i-1}} \cup \{x\} \quad \text{if} \quad M_{T_i} \leq \tfrac{c(L_{T_{i-1}}, x)}{H}, \\ U_{T_i} &:= U_{T_{i-1}} \qquad\quad \text{otherwise.} \\ L_{T_i} &:= L_{T_{i-1}} \cup \{x\} \quad \text{if} \quad M_{T_i} \leq \tfrac{c(U_{T_{i-1}}, x)}{H}, \\ L_{T_i} &:= L_{T_{i-1}} \qquad\quad \text{otherwise.} \end{aligned} \tag{C.3.3}$$

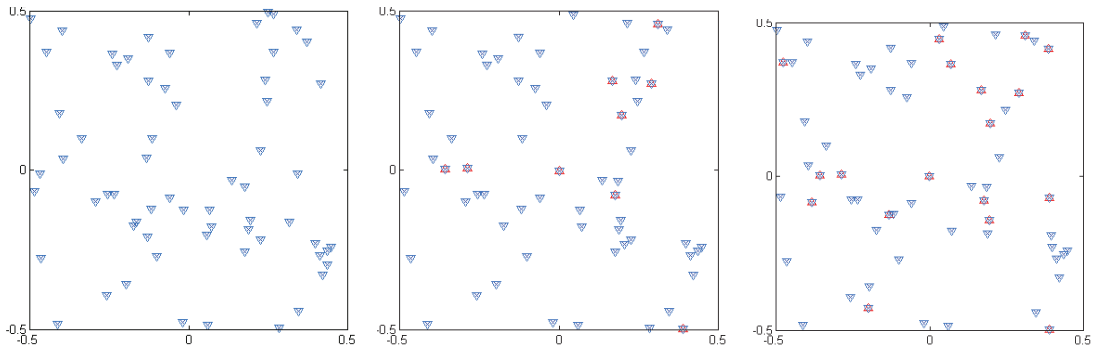
From Remark C.2.3, we know that $c(L_{T_{i-1}}, x) \geq c(U_{T_{i-1}}, x)$.
On the other hand, if there is a death of point $x \in D_{T_{i-1}}$ at time T_i, we configure: $U_{T_i} := U_{T_{i-1}} \backslash \{x\}$ and $L_{T_i} := L_{T_{i-1}} \backslash \{x\}$ respectively.

Suppose that the process (D, M) is stationary in time and $\hat{X}^n$, L^n and U^n for $\{n = 1, 2, 4, \ldots\}$ are derived adaptively from (D, M) satisfying (C.3.1) and (C.3.2), then the following proposition follows immediately from the dominated convergence theorem, see [41].

Proposition C.3.2. *Let $N = \inf\{n \in \{\frac{1}{2}, 1, 2, \cdots\} : L_0^n = U_0^n\}$, and set $L_{-n}^n = \emptyset$ and $U_{-n}^n = D_{-n}$. If as t tends to infinity, $\hat{X}_t$ converges weakly to an equilibrium distribution π and the probability of D visiting $\emptyset$ in the time interval $[0, t]$ converges to 1, then almost surely $N < \infty$ and $L_0^N = U_0^N$ follows the equilibrium distribution π.*

C.3.2 Simulation of determinantal point processes

In this subsection, we generalize the idea of Kendall and Møller [41] by relaxing the condition of compactness. As defined above, we denote a birth and death process with parameters B and d as $\mathrm{BDP}(B, d)$.



(a) at time T_0 (b) at time T_{25} (c) at time T_{50}

Figure C.1: CFTP simulations for Gaussian model DPP with $\rho = 50$ and $\alpha = 0.04$, respectively at time T_i, the i-th jump time from time $t = -n$. Notations: " $\cdot$ " $:= D_t$, "∇" $:= U_t$ and "Δ"(red) $:= L_t$

Theorem C.3.3. *Given a DPP X defined on a Polish space E w.r.t. a Radon measure ν. Suppose that the dominating process D is* $\mathrm{BDP}(\int_E J(x,x)d\nu(x), 1)$ *initiated with a Poisson point process (PPP) with intensity measure:*

$$\int_E J(x,x)d\nu(x) < \infty \tag{C.3.4}$$

Suppose further, the marking process $M_{T_i} \sim$ Unif $(0,1)$, i.i.d. for each i and independent of D. Then the process (D, M) is stationary in time and almost surely $N < \infty$ and $L_0^N = U_0^N$ follows the distribution of DPP X.

Proof. Stationarity of the process (D, M) follows immediately as the process D started with its equilibrium distribution, i.e. a $\mathrm{PPP}(J(x,x)d\nu(x))$.

Following, the probability of D visiting $\emptyset$ in the time interval $[0,t]$ converges to 1 as t tends to infinity. Consequently, from Proposition C.3.2 we have $L_0^N = U_0^N$ follows the distribution of DPP X. □

Algorithm 10 Simulation of determinantal point process

```
Sample D_0 from PPP (∫_E J(x,x)dν(x))
n ← 1/2;
while TRUE do
  D ← BackwardExtend(D, n);
  [L, U] ← Coupling(D);
  if L_0 == U_0 then
    return L_0
  else
    n ← n * 2;
  end if
end while
```

Algorithm 11 BackwardExtend(D, n)

$j \leftarrow 0$;
$T(0) \leftarrow n/2$; $\{T(0) \leftarrow 0$ if $n = 1/2\}$
$\tilde{D}_{T(0)} \leftarrow D_{-n/2}$; $\{\tilde{D}_{T(0)} \leftarrow D_0$ if $n = 1/2\}$
while $T(j) \leq n$ **do**
  $T(j+1) \leftarrow T(j) - \log(\text{Uniform}(0,1))/(\int_E J(x,x)d\nu(x) + |\tilde{D}_{T(j)}|)$;
  if $\text{Uniform}(0,1) \leq (\int_E J(x,x)d\nu(x))/(\int_E J(x,x)d\nu(x) + |\tilde{D}_{T(j)}|)$ **then**
    $x \leftarrow$ uniform random point in $\Lambda \backslash \tilde{D}_{T(j)}$;
    $\tilde{D}_{T(j+1)} \leftarrow \tilde{D}_{T(j)} \cup x$;
  else
    $x \leftarrow$ uniform random point in $\tilde{D}_{T(j)}$;
    $\tilde{D}_{T(j+1)} \leftarrow \tilde{D}_{T(j)} \backslash x$;
  end if
  $j \leftarrow j + 1$;
end while
$D_{-t} \leftarrow \tilde{D}_t$ for all $t : n/2 < t \leq n$
return D

Algorithm 12 [41] Coupling(D)

$L_{-n} \leftarrow \emptyset$;
$U_{-n} \leftarrow D_{-n}$;
for $T_i \leftarrow$ each jump times $T_1 < T_2 < \cdots$ of D in $]-n : 0]$ **do**
  if $D_{T_i} \leftarrow D_{T_{i-1}} \cup x$ **then**
    $u \leftarrow M_T$;
    $[L_{T_i}, U_{T_i}] \leftarrow \text{AddBirth}(L_{T_{i-1}}, U_{T_{i-1}}, x, u)$;
  else
    $x \leftarrow D_{T_{i-1}} \backslash D_{T_i}$;
    $L_{T_i} \leftarrow L_{T_{i-1}} \backslash x$;
    $U_{T_i} \leftarrow U_{T_{i-1}} \backslash x$;
  end if
end for
return $[L, U]$

In the pseudocode, we begin with setting $n = 1/2$ (in later Section C.4, we will show that we can replace $n = 1/2$ with $\log \int_E J(x,x)d\nu(x))$ and construct D backwards in time to $-n$. Instead of reversing the birth and death rate, the process D is a Glauber process with equilibrium measure $\int_E J(x,x)d\nu(x)$ and hence we can simulate $\tilde{D}_n$ as BDP$(\int_E J(x,x)d\nu(x), 1)$ forwards in time to time n, initiated with state $\tilde{D}_0 = D_0 = PPP(J(x,x)d\nu(x))$. Let $D_{-t} = \tilde{D}_t$ for all $t : 0 \leq t \leq n$. We initiate the processes L and U from time $-n$ with $L_{-n} = \emptyset$ and $U_{-n} = D_{-n}$ respectively, such that (C.3.1) is satisfied. The implementation of AddBirth$(L_{T_{i-1}}, U_{T_{i-1}}, x, u)$ in Algorithm 12 follows directly from: if there is a birth of point $x \in \Lambda \backslash D_{T_{i-1}}$ at time T_i, we set:

$$\begin{aligned} U_{T_i} &:= U_{T_{i-1}} \cup \{x\} \quad \text{if} \quad M_{T_i} \leq \tfrac{\int_E c(L_{T_{i-1}},x)d\nu(x)}{\int_E J(x,x)d\nu(x)}, \\ U_{T_i} &:= U_{T_{i-1}} \qquad \text{otherwise.} \\ L_{T_i} &:= L_{T_{i-1}} \cup \{x\} \quad \text{if} \quad M_{T_i} \leq \tfrac{\int_E c(U_{T_{i-1}},x)d\nu(x)}{\int_E J(x,x)d\nu(x)}, \\ L_{T_i} &:= L_{T_{i-1}} \qquad \text{otherwise.} \end{aligned} \tag{C.3.5}$$

and if there is a death of point $x \in D_{T_{i-1}}$ at time T_i, we configure: $U_{T_i} := U_{T_{i-1}} \backslash \{x\}$ and $L_{T_i} := L_{T_{i-1}} \backslash \{x\}$ respectively. In the algorithm, the coalescence of processes L_t and U_t, and their convergence to the target distribution is assured by Theorem C.3.3.

C.4 Running time

In the previous sections, we have been concerned with the simulation of a DPP defined on a Polish space E. Following, we will provide a bound for the running time of the algorithm through the asymptotic behavior of a $M/M/\infty$ queue. We defined the process $G_t := |U_t \backslash L_t|$ for all $t \in \mathbb{R}$.

The coalescence time of L and U (equivalently the running time of the algorithm) is equal to the hitting time of G_t to 0. From (C.3.3), a birth occur in G_t if and only if there is a birth in U_t but not in L_t and a death occur in G_t if and only if a death occur in D_t and the dead point x is in U_t but not in L_t. We obtain the arrival rate, λ^G and service rate, μ^G of G_t as following:

$$\lambda^G = \int_E c(L_t, x) - c(U_t, x)d\nu(x) \tag{C.4.1}$$

$$\mu_t^G = \tfrac{|G_t|}{|U_t|} \tag{C.4.2}$$

By definition of G_t, $|D_t|$ is an upper bound for G_t for all t. Denote $Z_t := |D_t|$, and suppose that the process Z_t initiated at time $t = t_0$ and we have a sufficiently large $|D_{t_0}| = z$, then after an exponential time, the process goes to size $z+1$ with probability $\frac{\lambda}{\lambda + z\mu}$ and $z-1$ with probability $\frac{z\mu}{\lambda + z\mu}$. Hence, the next jump is very likely to be a death. We can roughly approximate the order of hitting time by

$$\sum_{i=1}^{z} \frac{1}{\lambda + i\mu} \sim \frac{\log z}{\mu}$$

A rigorous approach is provided in [71], as the following Proposition C.4.1.

Proposition C.4.1. *Given a $M/M/\infty$ queue Z_t with arrival rate λ and service rate μ (initiated at t_0, with $Z_{t_0} = z$), the hitting time T_0^D of 0 is of the order $\log z$. Precisely:*

$$\lim_{z \to \infty} \Pr\left(\left| \frac{T_0^D}{\log z} - \frac{1}{\mu} \right| \geq \varepsilon \right) = 0.$$

On the other hand, the hitting time of G_t to 0 is at least lower bounded by the time where all the initial points in $U_{t_0}\backslash L_{t_0}$ perished, accordingly we have the following Proposition.

Proposition C.4.2. *Given a $M/M/\infty$ queue G_t with arrival rate λ^G and service rate μ_t^G (initiated at t_0, with $G_{t_0} = z$ and $\mu_{t_0}^G = 1$), the expected hitting time $\mathrm{E}[T_0]$ of 0 is lower bounded:*

$$\log z \quad \leq \quad \sum_{i=1}^{z} \frac{1}{i} \quad = \quad \mathrm{E}[T_0] \tag{C.4.3}$$

Proof. Given $\mu_{t_0}^G = 1$, each of the initial point in the configuration $U_{t_0}\backslash L_{t_0}$ have living period i.i.d. Exponential(1). It is not difficult to show that the expected value of the maximum of z exponential random variable, x_i with parameter 1 is:

$$\mathrm{E}[\max(x_i)] = \sum_{i=1}^{z} \frac{1}{i} \qquad \text{(harmonic series)}.$$

Hence, (C.4.3) is proved. □

From Propositions C.4.1 and C.4.2, given $N = \inf\{n \in \{\frac{1}{2}, 1, 2, \cdots\} : L_0^n = U_0^n\}$ and the size of the initial configuration of the process G_{-N}^N equal to z, we have the following:

— the expected coalescence time, $\mathrm{E}[T_0]$ is lower bounded by $\log z$.

— the upper bound of the coalescence time, T_0^D converges in probability to $\log z$ as z tends to infinity.

Consequently, we have the following theorem.

Theorem C.4.3. *The coalescence time, T_0 is asymptotically of the order $\log z$, i.e.*

$$\lim_{z\to\infty} \Pr\left(\left|\frac{T_0}{\log z} - \frac{1}{\mu}\right| \geq \varepsilon\right) = 0.$$

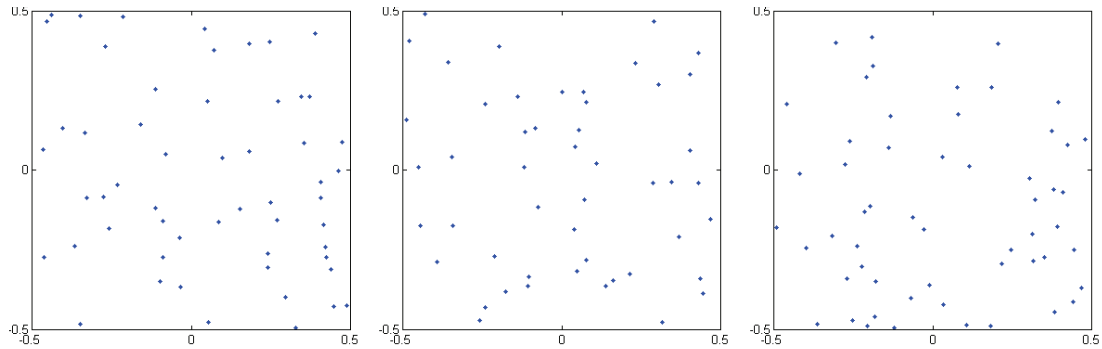
Proof. We define a lower process $M/M/\infty$ queue with initial size z, arrival rate 0 and service rate 1. By Proposition C.4.1, as z tends to infinity, the hitting time T_0^L is of the order $\log z$. Given any constant $c \geq 0$, we know that $\Pr(T_0^L < c) \geq \Pr(T_0 < c)$.

By the definition of G_t, we have $T_0 \leq T_0^D$ almost surely. Hence, for all $\epsilon > 0$,

$$\begin{aligned}
\lim_{z\to\infty} \Pr\left(\left|\frac{T_0}{\log z} - 1\right| \geq \epsilon\right) &= \lim_{z\to\infty} \left(\Pr\left(\frac{T_0}{\log z} \geq 1-\epsilon\right) + \Pr\left(\frac{T_0}{\log z} \leq 1+\epsilon\right)\right) \\
&\leq \lim_{z\to\infty} \left(\Pr\left(\frac{T_0^D}{\log z} \geq 1-\epsilon\right) + \Pr\left(\frac{T_0^L}{\log z} \leq 1+\epsilon\right)\right) \\
&= 0
\end{aligned}$$

□

Heuristically, we shall see that the running time of the algorithm is of the order $O(\log \int_E J(x,x)d\nu(x))$ in Section C.5.



(a) Gaussian Model (b) Matérn Model ($\nu = 5$) (c) Cauchy model ($\nu = 5$)

Figure C.2: Configurations of stationary DPPs simulated from CFTP algorithm. $\rho = 50$ and $\alpha = \alpha_{\max}/2$ for all the 3 models.

C.5 Application to examples in stationary models

Point processes and random measures that are invariant under shifts in a d-dimensional Euclidean space $\mathbb{R}^d$ play a vital role in applications and development of the general theories. Accordingly in this section, we will be focusing on stationary DPP models defined on a compact set $\Lambda = [-1/2, 1/2]^2$ in $\mathbb{R}^2$, where we denote $l \in \Delta := \{x - y : x, y \in \Lambda\}$ and

$$\mathbb{K}_0(l) := \mathbb{K}(x, y).$$

Suppose we are given a DPP kernel, it would be ideal if we could explicitly compute the Papangelou intensity (upper bound) in Lemma C.2.2, however (see [28] and [44]) there are only known in a few simple models where the Papangelou intensity can be computed from the spectral representation. In the sequel, we will approximate the kernel in (C.2.1) by applying Fourier expansion to obtain the Papangelou intensity.

Consider the following orthonormal Fourier basis in $L^2(\Delta)$:

$$\varphi_k^{\Delta}(l) = e^{2\pi i k \cdot (l)}, \quad k \in \mathbb{Z}^2, l \in \Delta \tag{C.5.1}$$

where $k \cdot l$ is the dot product of the vectors. We have the Fourier expansion of $\mathbb{K}_0(l)$ as:

$$\mathbb{K}_0(l) = \sum_{k \in \mathbb{Z}^2} \lambda_k^{\Delta} \varphi_k^{\Delta}(l).$$

equivalently:

$$\mathbb{K}(x, y) = \sum_{k \in \mathbb{Z}^2} \lambda_k^{\Delta} \varphi_k^{\Delta}(x) \overline{\varphi_k^{\Delta}(y)},$$

where the Fourier coefficients:

$$\begin{aligned} \lambda_k^{\Delta} &= \int_{\Delta} \mathbb{K}_0(l) e^{-2\pi i k \cdot r} dl \\ &\approx \int_{\mathbb{R}^2} \mathbb{K}_0(r) e^{-2\pi i k \cdot r} dr \quad =: \quad \lambda_k \end{aligned}$$

As $\mathbb{K}_0(l) \approx 0$ when $|l| > 1$, we have Fourier transform as an approximation for the Fourier coefficient.

Corresponding to [44], let ρ defined the intensity of the DPP and we apply the CFTP simulations on the following three stationary models and their Fourier transform respectively:

1. Gaussian model: for $\rho \geq 0$ and $0 < \alpha \leq \sqrt{\frac{1}{\pi\rho}}$

$$\mathbb{K}(x,y) = \rho \exp\left(-\frac{1}{\alpha^2}\|x-y\|^2\right) \tag{C.5.2}$$

$$\lambda_k = \pi\alpha^2\rho e^{-\pi^2\alpha^2\|k\|^2} \tag{C.5.3}$$

2. Matérn model: for $\rho \geq 0$, $\nu > 0$ and $0 < \alpha \leq \sqrt{\frac{1}{4\pi\nu\rho}}$

$$\mathbb{K}(x,y) = \rho\frac{2^{1-\nu}}{\Gamma(\nu)\alpha^\nu}\|x-y\|^\nu \mathbf{K}_\nu(\frac{1}{\alpha}\|x-y\|) \tag{C.5.4}$$

$$\lambda_k = 4\pi\alpha^2\rho\frac{\nu}{(1+4\pi^2\alpha^2\|k\|^2)^{1+\nu}} \tag{C.5.5}$$

3. Cauchy model (generalized): for $\rho \geq 0$, $\nu > 0$ and $0 < \alpha \leq \sqrt{\frac{\nu}{\pi\rho}}$

$$\mathbb{K}(x,y) = \frac{\rho}{(1+\frac{1}{\alpha^2}\|x-y\|^2)^{1+\nu}} \tag{C.5.6}$$

$$\lambda_k = \frac{2^{1-\nu}\pi\alpha^2\rho}{\Gamma(1+\nu)}\|2\pi\alpha k\|^\nu \mathbf{K}_\nu(\|2\pi\alpha k\|) \tag{C.5.7}$$

where $\mathbf{K}_\nu$ is the modified Bessel function of the second kind. The integral representation of the function is:

$$\mathbf{K}_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}dt$$

As z tends to 0, we have $\mathbf{K}_\nu(z)$ tends to infinity and $\lim_{z\to 0} z^\nu \mathbf{K}_\nu(z) = \frac{\Gamma(\nu)}{2^{1-\nu}}$. In Matérn model, we have $\mathbb{K}(x,x) = \rho$, and in Cauchy model, we have $\lambda_0 = \frac{\pi\alpha^2\rho}{\nu}$. See other representations of $\mathbf{K}_\nu(z)$ in [36].

From Lemma C.2.2 and (C.2.4), we set the upper bound of the Papangelou intensity:

$$H = J_\Lambda(x,x) = \sum_{k\in\mathbb{Z}^2}\frac{\lambda_k}{1-\lambda_k} \tag{C.5.8}$$

In practice we are unable to compute the infinite sum of the Fourier expansions. From (C.2.2), a rule of thumb will be choosing a constant N large enough such that that:

$$\sum_{-N\leq i,j\leq N}\lambda_{i,j}^\Delta \approx \rho \tag{C.5.9}$$

For the given models, we compute their pair correlation function $g(r)$ to investigate the reliability of the simulations, where $r := \|x-y\|^2$. Given any 2 points x and y in a configuration, their pair correlation function is given as:

$$g(r) := \frac{\rho_2(x,y)}{\rho_1(x)\rho_1(y)} = 1 - \frac{\mathbb{K}(x,y)\mathbb{K}(y,x)}{\mathbb{K}(x,x)\mathbb{K}(y,y)}$$

and respectively for:

1. Gaussian model:

$$g(r) \quad = \quad 1 - e^{-2(r/\alpha)^2}, \tag{C.5.10}$$

2. Matérn model:

$$g(r) \quad = \quad 1 - [2^{1-\nu}(\frac{r}{\alpha})^{\nu}\mathbf{K}_{\nu}\left(\frac{r}{\alpha}\right)/\Gamma(\nu)]^2, \tag{C.5.11}$$

3. Cauchy model:

$$g(r) \quad = \quad 1 - (1 + (r/\alpha)^2)^{-2\nu-2}. \tag{C.5.12}$$

In this paper, we fixed N such that the sum is at least 99.9% of ρ. In Figure C.3, we compare the distribution of the size of DPP simulated with its *actual* distribution given by a Poisson-binomial distribution, see [34]. Here, we use the term *actual* for the truncated Fourier expansion to approximate the kernel. From [26], the probability mass function (PMF) of the Poisson-binomial distribution can be written in the form of discrete Fourier transform as:

$$\Pr(|X| = n) = \frac{1}{N+1}\sum_{k=0}^{N} e^{\frac{-2\pi ikn}{N+1}} \prod_{m=1}^{N}\left(p_m e^{\frac{2\pi ik}{N+1}} + (1-p_m)\right)$$

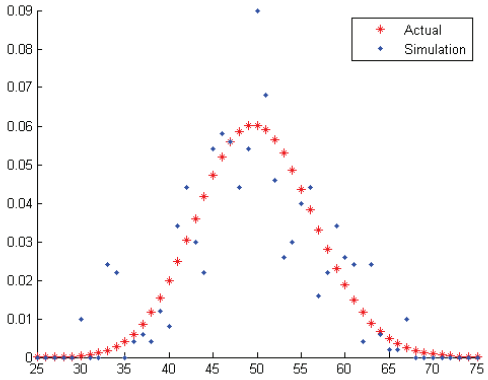
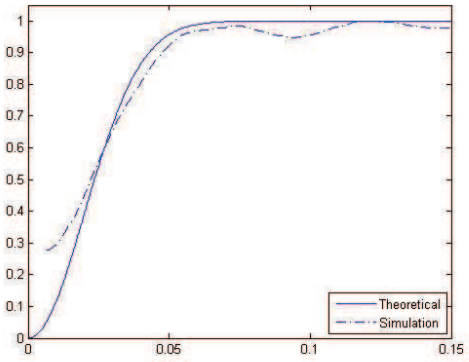


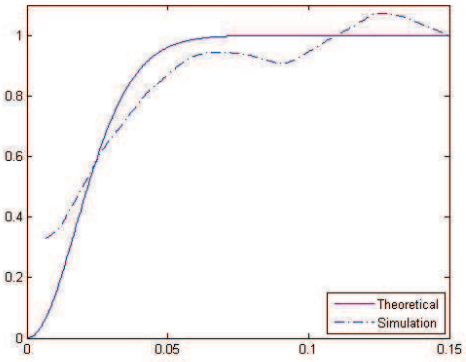
Figure C.3: PMF of 500 Gaussian models with $\rho = 50$ and $\alpha = 0.04$.

Figure C.4 is a comparison of the theoretical pair correlation function $g(r)$ with the simulated DPP results for $\rho = 50$ and α is fixed to be $\alpha_{\max}/2$. For both Matérn Model and Cauchy Model, $\nu = 5$. With regards to the the error in $g(r)$ when r tends to 0, we lost some 'harmonic' in the Fourier approximations when computing the Papangelou intensity.

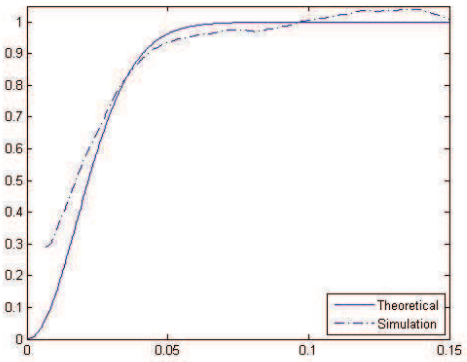
Figure C.5(a) and Figure C.5(b) show the distributions of the coalescence time of L_t and U_t and the stopping time of the algorithm for 500 Gaussian model DPP simulations with $\rho = 50$. The stopping time refer to the time $-n$ required to simulate backwards such that coalescence of L_t and U_t occur at time $t = 0$. Heuristically, we have shown that the stopping time is of order $O(\log H|\Lambda|)$, $(H \approx 57.5)$.



(a) Gaussian Model

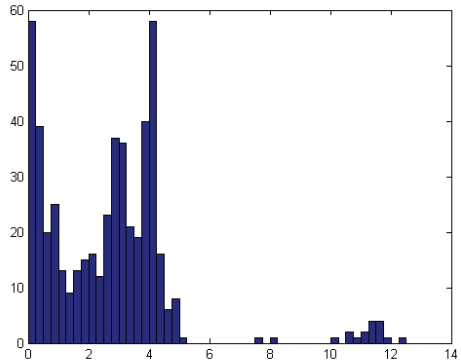


(b) Matérn Model

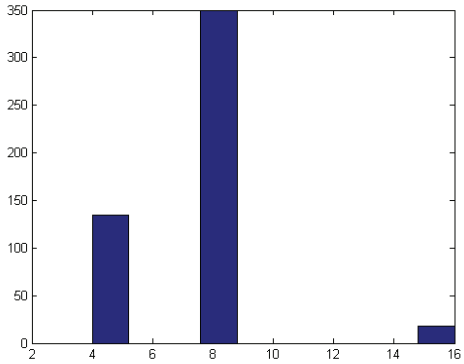


(c) Cauchy model

Figure C.4: Pair Correlation Function.



(a) Coalescence time of L_t and U_t



(b) Stopping time

Figure C.5: Histograms of coalescence time of L_t and U_t and the stopping time on 500 Gaussian models with $\rho = 50$ and $\alpha = 0.04$.

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Analyse stochastique de processus ponctuels : Au-delà du processus de Poisson

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RESUME : Les processus déterminantaux ont généré de l'intérêt dans des domaines très divers, tels que les matrices aléatoires, la théorie des processus ponctuels, ou les réseaux. Dans ce manuscrit, nous les considérons comme un type de processus ponctuel, c'est-à-dire comme un groupement de points aléatoires dans un espace très général. Ainsi, nous avons accès à une grande variété d'outils provenant de la théorie des processus ponctuels, ce qui permet une analyse précise d'un grand nombre de leur propriétés.

Nous commençons par poser les bases probabilistes selon lesquelles nous allons travailler dans ce manuscrit. Nous insistons tout particulièrement sur les propriétés plus subtiles de l'intensité de Papangelou, qui sera centrale dans de nombreux chapitres.

Puis, nous passons à une analyse des processus déterminantaux d'un point de vue applicatif. Nous proposons ainsi différentes méthodes pour leur simulation dans un cadre général. Nous présentons une série de modèles dérivés du processus de Ginibre, et qui se trouvent être très utiles dans les applications.

Troisièmement, nous introduisons un gradient différentiel sur l'espace des processus ponctuels. Grâce à des outils puissants de la théorie générale des formes de Dirichlet, nous montrons une formule d'intégration par parties pour un processus ponctuel général, et prouvons l'existence de diffusions correctement associées à ces processus. Nous sommes en mesure d'appliquer ces résultats aux processus déterminantaux, ce qui mènera à une caractérisation de ces diffusions en termes d'équations différentielles stochastiques.

Enfin, nous nous intéressons au gradient différence. Dans un certain sens, nous définissons alors une intégrale de Skohorod qui satisfait une formule d'intégration par parties, c'est-à-dire que son adjoint est le gradient différence. Une application à l'étude d'une transformation aléatoire du processus ponctuel est présentée, dans laquelle nous caractérisons la distribution du processus ponctuel transformé sous certaines conditions.

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